

ANALYTIC AND REIDEMEISTER TORSION FOR REPRESENTATIONS IN FINITE TYPE HILBERT MODULES

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ABSTRACT. For a closed Riemannian manifold (M, g) we extend the definition of analytic and Reidemeister torsion associated to an orthogonal representation of $\pi_1(M)$ on a \mathcal{A} -Hilbert module \mathcal{W} of finite type where \mathcal{A} is a finite von Neumann algebra.

If (M, \mathcal{W}) is of determinant class we prove, generalizing the Cheeger-Müller theorem, that the analytic and Reidemeister torsion are equal. In particular, this proves the conjecture that for closed Riemannian manifolds with positive Novikov-Shubin invariants, the L_2 -analytic and L_2 -Reidemeister torsions are equal.

0. INTRODUCTION

The purpose of this paper is to prove the equality of L_2 -analytic and L_2 -Reidemeister torsion. Both torsions are numerical invariants defined for closed manifolds of determinant class, in particular for closed manifolds with positive Novikov-Shubin invariants. For these manifolds their equality has been conjectured by Carey, Mathai, Lott, Lück, Rothenberg and others (cf e.g. [LL, conjecture 9.7]). It is implicit in [Lü3](cf [BFK3]) that any closed manifold whose fundamental group is residually finite is of determinant class.

The interest of the conjecture comes, among other issues, from the geometric significance of the L_2 -analytic torsion and the fact that sometimes the L_2 -Reidemeister torsion can be computed numerically in a very efficient way. Indeed, if M is a closed hyperbolic manifold of dimension 3, the L_2 -analytic torsion coincides, up to a factor $-1/3\pi$, with the hyperbolic volume.

We establish the conjecture by proving a more general result: Given a closed Riemannian manifold (M, g) , we extend the notion of analytic and Reidemeister torsions to orthogonal representations of the fundamental group $\pi_1(M)$ on a \mathcal{A} -Hilbert module \mathcal{W} of finite type where \mathcal{A} is a finite von Neumann algebra, and prove the equality of the two torsions when (M, \mathcal{W}) is of determinant class. We point out that in the case where \mathcal{A} is \mathbb{R} , we obtain a

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new proof of the well known result due to, independently, Cheeger [Ch] and Müller [Mü]. From the analytic point of view, the main difficulty comes from the fact that the Laplacians, associated to such representations, may have continuous spectrum. In particular, 0 might be in the essential spectrum.

In order to formulate our results more precisely, we introduce the following notations.

Let M be a closed smooth manifold. A generalized triangulation of M is a pair $\tau = (h, g')$ with the following properties:

(T1) $h : M \rightarrow \mathbb{R}$ is a smooth Morse function which is self-indexing ($h(x) = \text{index}(x)$ for any critical point x of h);

(T2) g' is a Riemannian metric so that $\text{grad}_{g'} h$ satisfies the Morse-Smale condition (for any two critical points x and y of h , the stable manifold W_x^+ and the unstable manifold W_y^- , with respect to $\text{grad}_{g'} h$, intersect transversely);

(T3) in a neighborhood of any critical point of h one can introduce local coordinates such that, with q denoting the index of this critical point,

$$h(x) = q - (x_1^2 + \dots + x_q^2)/2 + (x_{q+1}^2 + \dots + x_d^2)/2$$

and the metric g' is Euclidean in these coordinates.

The unstable manifolds W_x^- provide a partition of M into open cells where W_x^- is an open cell of dimension equal to the index of x . The name “generalized triangulation” for the pair (h, g') is justified as a generalized triangulation can be viewed as a generalization of a simplicial triangulation.¹

Let (M, g) be a closed Riemannian manifold with infinite fundamental group $\Gamma = \pi_1(M)$ and let $\tau = (h, g')$ be a generalized triangulation. Note that Γ is countable. Let $p : \tilde{M} \rightarrow M$ be the universal covering of M and denote by \tilde{g} and $\tilde{\tau} = (\tilde{h}, \tilde{g}')$ the lifts of g and τ on \tilde{M} . The Laplace operator Δ_q acting on compactly supported, smooth q -forms on \tilde{M} is essentially selfadjoint. Its closure, also denoted by Δ_q , is therefore selfadjoint; it is defined on a dense subspace of the L_2 -completion of the space of smooth forms with compact support with respect to the scalar product induced by the metric \tilde{g} . Observe that Δ_q is Γ -equivariant and nonnegative. We can therefore define the spectral projectors $Q_q(\lambda)$ of Δ_q corresponding to the interval $(-\infty, \lambda]$. They are Γ -equivariant and admit a Γ -trace which we denote by $N_q(\lambda)$.

Let $\mathcal{C}^q(\tilde{\tau}) := l^2(\text{Cr}_q(\tilde{h}))$ where $l^2(\text{Cr}_q(\tilde{h}))$ denotes the Hilbert space of l_2 -summable, real-valued sequences indexed by the countable set $\text{Cr}_q(\tilde{h})$ of critical points of \tilde{h} of index q . The left action of Γ on $\text{Cr}_q(\tilde{h})$ makes $l^2(\text{Cr}_q(\tilde{h}))$ the underlying Hilbert space of an orthogonal Γ -representation. The intersections of the stable and the unstable manifolds of $\text{grad}_{\tilde{g}} \tilde{h}$ induce a bounded, Γ -equivariant, linear map

$$\delta_q : \mathcal{C}^q(\tilde{\tau}) \rightarrow \mathcal{C}^{q+1}(\tilde{\tau}).$$

¹Given a smooth simplicial triangulation τ_{sim} , one can construct a generalized triangulation $\tau = (h, g')$ so that the unstable manifolds W_x^- corresponding to $\text{grad}_{g'} h$, with x a critical point of h , are the open simplexes of τ_{sim} (cf. [Po]).

Let δ_q^* be the adjoint of δ_q and introduce

$$\Delta_q^{\text{comb}} := \delta_q^* \cdot \delta_q + \delta_{q-1} \cdot \delta_{q-1}^*.$$

Observe that Δ_q^{comb} is a Γ -equivariant, bounded, nonnegative, selfadjoint operator on $C^q(\tilde{\tau})$. We can therefore define the spectral projectors $Q_q^{\text{comb}}(\lambda)$ of Δ_q^{comb} corresponding to the interval $(-\infty, \lambda]$. These projectors are Γ -equivariant and thus admit a Γ -trace, which we denote by $N_q^{\text{comb}}(\lambda)$ (cf section 1).

We say that

- (1) (M, g) is of a - *determinant* class if $-\infty < \int_{0+}^1 (\log \lambda) dN_q(\lambda)$ for all q .
- (2) (M, τ) is of c - *determinant* class if $-\infty < \int_{0+}^1 (\log \lambda) dN_q^{\text{comb}}(\lambda)$ for all q .

Here \int_{0+}^1 denotes the Stieltjes integral on the half open interval $(0, 1]$. The following result can be derived from work of Gromov-Shubin [GS] (cf also [Ef]).

Proposition 1. ([Ef], [GS]) *Let M be a closed manifold equipped with a Riemannian metric g and a generalized triangulation τ . Then:*

- (1) *(M, g) is of a - determinant class iff (M, τ) is of c - determinant class.*
- (2) *Let (M', τ') be another manifold with generalized triangulation τ' . If M and M' are homotopy equivalent, then (M, τ) is of c - determinant class iff (M', τ') is.*

Definition. *A compact manifold M is of determinant class if for some generalized triangulation τ (and then for any), (M, τ) is of c - determinant class.*

If M is of determinant class the logarithm of the ζ -regularized determinant, $\log \det_N \Delta_q$, is a finite real number for all q and one can introduce the L_2 - analytic torsion T_{an} :

$$\log T_{an} := \frac{1}{2} \sum_q (-1)^{q+1} q \log \det_N \Delta_q.$$

Similarly, if M is of determinant class, $\log \det_N \Delta_q^{\text{comb}}$ is a finite real number for all q and one can define the combinatorial torsion:

$$\log T_{\text{comb}} := \frac{1}{2} \sum_q (-1)^{q+1} q \log \det_N \Delta_q^{\text{comb}}.$$

To define the L_2 -Reidemeister torsion, T_{Re} , it remains to introduce its metric part. Notice that $\text{Null}(\Delta_q)$ consists of smooth forms and that integration of smooth q - forms over a smooth q -chain induces (cf a theorem by Dodziuk [Do]) an isomorphism $\theta_q^{-1} : \text{Null}(\Delta_q) \rightarrow \overline{H}^q(C^*(\tilde{\tau}), \delta_*) = \text{Null}(\Delta_q^{\text{comb}})$ of $\mathcal{N}(\Gamma)$ -Hilbert modules where $\mathcal{N}(\Gamma)$ is the von Neumann algebra associated to Γ . Define

$$\log \text{Vol}_N(\theta_q) := \frac{1}{2} \log \det_N(\theta_q^* \theta_q)$$

where we used that $\det_N(\theta_q^* \theta_q) > 0$ as $(\theta_q^* \theta_q)$ is a selfadjoint, positive, bounded, Γ -equivariant operator on the Γ - Hilbert space $\text{Null}(\Delta_q^{\text{comb}})$ whose spectrum is bounded away from 0. As a consequence (cf section 1) $\log \det_N(\theta_q^* \theta_q)$ is a well defined real number and one may introduce

$$\log T_{\text{met}} := \frac{1}{2} \sum_q (-1)^q \log \det_N(\theta_q^* \theta_q).$$

Combining the above definitions we define the L_2 -Reidemeister torsion T_{Re}

$$\log T_{\text{Re}} = \log T_{\text{comb}} + \log T_{\text{met}}.$$

The concepts of L_2 -analytic and L_2 -Reidemeister torsion were considered by Novikov-Shubin in 1986 [NS1] (cf.also later work by Lott [Lo], Lück-Rothenberg [LR], Mathai [Ma] and Carey-Mathai [CM]). The main objective of this paper is to prove the following

Theorem 1. *Let M be a closed manifold of determinant class of odd dimension d . Then, for any Riemannian metric g and for any generalized triangulation τ , both T_{an} and T_{Re} are positive real numbers and*

$$T_{\text{an}} = T_{\text{Re}}.$$

We point out that the corresponding result for manifolds of even dimension is trivial as both torsions are equal to 1.

Before explaining the ideas of the proof of Theorem 1 let us make a few remarks:

1) Lott-Lück have conjectured (cf [LL], conjecture 9.2) that all compact manifolds have positive Novikov-Shubin invariants and, therefore, are of determinant class. The conjecture has been verified for many compact manifolds and in particular for all compact manifolds whose fundamental group is free or free abelian.

2) By assigning to each compact Riemannian manifold (M, g) with M of determinant class the L_2 -(analytic=Reidemeister) torsion if the fundamental group $\pi_1(M)$ is infinite and the usual (analytic=Reidemeister) torsion if $\pi_1(M)$ is finite one obtains a numerical invariant $T(M, g)$, which satisfies the product formula

$$\log T(M_1 \times M_2, g_1 \times g_2) = \chi(M_2) \log T(M_1, g_1) + \chi(M_1) \log T(M_2, g_2),$$

and for any n -sheeted covering (\tilde{M}, \tilde{g}) of (M, g) satisfies

$$\log T(\tilde{M}, \tilde{g}) = n \log T(M, g).$$

Here $\chi(M)$ denotes the Euler-Poincaré characteristic of M . For compact manifolds with trivial L_2 Betti numbers, in particular for manifolds of the homotopy type of a mapping torus, this invariant is independent of the metric and is in fact a homotopy type invariant. This invariant was calculated for a large class of 3-dimensional manifolds ; its logarithm is zero for Seifert manifolds (cf [LR]) and $(-1/3\pi) \text{Vol}(M, g)$ for a hyperbolic manifold (M, g) , (cf [Lo]). The calculation in [LR] was done for the Reidemeister torsion and in [Lo]

for the analytic torsion.

3) Recently W. Lück ([Lü2]) found an algorithm to calculate the L_2 -Reidemeister torsion of a 3-dimensional hyperbolic manifold in terms of a balanced presentation of its fundamental group. By Theorem 1 and by Remark 2) above the algorithm also calculates the hyperbolic volume.

Rather than viewing the above theorem as an L_2 -version, we derive Theorem 1 as a particular case of a generalization of the Cheeger-Müller theorem. This generalization concerns the extension of the analytic and Reidemeister torsion associated to a closed Riemannian manifold and a finite dimensional orthogonal representation of Γ to orthogonal \mathcal{A} -representation of Γ which are of finite von Neumann dimension where \mathcal{A} is a finite von Neumann algebra (cf [Si] for a similar approach in connection with an L_2 -index theorem). A representation of this type is called an $(\mathcal{A}, \Gamma^{op})$ -Hilbert module of finite type.

In order to formulate this generalization we must introduce (cf section 2) a calculus of elliptic pseudodifferential \mathcal{A} -operators acting on sections of a bundle of \mathcal{A} -Hilbert modules of finite type over a compact manifold (typically the spectrum of such an operator is no longer discrete) and develop a theory of regularized determinants for (nonnegative) elliptic pseudodifferential \mathcal{A} -operators of positive order. Both ingredients are presented in section 2. The new difficulties in this theory come from the fact that 0 might be in the essential spectrum of an elliptic pseudodifferential \mathcal{A} -operator.

In order to prove this generalization we need a Mayer-Vietoris type formula and an asymptotic expansion for the logarithm of the determinant of elliptic operators with parameter; i.e. the extension of the results of [BFK2] to this calculus (cf section 3 and [Lee]).

Let us now describe the generalization of the above Theorem in more detail. Assume that A is an elliptic operator in the new calculus. For an angle θ and $\epsilon > 0$ introduce the solid angle

$$V_{\theta, \epsilon} := \{z \in \mathbb{C} : |z| < \epsilon\} \cup \{z \in \mathbb{C} \setminus 0 : \arg(z) \in (\theta - \epsilon, \theta + \epsilon)\}.$$

Definition. (1) θ is an Agmon angle for A , if there exists $\epsilon > 0$ so that

$$\text{spec}(A) \cap V_{\theta, \epsilon} = \emptyset.$$

(2) θ is a principal angle for A if there exists $\epsilon > 0$ so that

$$\text{spec}(\sigma_A(x, \xi)) \cap V_{\theta, \epsilon} = \emptyset$$

for all $(x, \xi) \in S_x^*M$ where S^*M denotes the cosphere bundle and $\sigma_A(x, \xi)$ is the principal symbol of A .

It is well known that (1) implies (2) in the above definition but the converse statement is not true. If, in addition, A is of order $m > 0$ and admits an Agmon angle, θ , one can define the regularized determinant, $\det_{\theta, N} A \in \mathbb{C}$. In the sequel, θ will always be

chosen to be π and we will drop the subscript π in $\det_{\pi,N}$. If A is of order $m > 0$, nonnegative and if $0 \in \text{spec}(A)$ then the ellipticity of A implies that the nullspace, $\text{Null}(A)$, is an \mathcal{A} -Hilbert module of finite von Neumann dimension, $\dim_N \text{Null}(A)$. Consider the 1-parameter family, $A + \lambda$, λ being the spectral parameter. For $\lambda > 0$, introduce the function $\log \det_N(A + \lambda) - \dim_N \text{Null}(A) \log \lambda$. We can view this function as an element in the vector space \mathbf{D} consisting of equivalence classes $[f]$ of real analytic functions $f : (0, \infty) \rightarrow \mathbb{R}$ with $f \sim g$ iff $\lim_{\lambda \rightarrow 0} (f(\lambda) - g(\lambda)) = 0$. The elements of \mathbf{D} represented by the constant functions form a subspace of \mathbf{D} which can be identified with \mathbb{R} , the space of real numbers.

Given a closed Riemannian manifold (M, g) , an arbitrary $(\mathcal{A}, \Gamma^{op})$ -Hilbert module of finite type, \mathcal{W} , and a generalized triangulation τ we define (cf section 4) $\log T_{\text{an}}(M, g, \mathcal{W})$ and $\log T_{\text{Re}}(M, g, \tau, \mathcal{W})$ as elements of \mathbf{D} . As above we consider the analytic resp. combinatorial Laplacians associated to (M, g, \mathcal{W}) resp. (M, τ, \mathcal{W}) and introduce the notion of a triple (M, g, \mathcal{W}) resp. (M, τ, \mathcal{W}) , of a -determinant, resp. of c -determinant class. Proposition 1 can be generalized to say that these two notions are equivalent and homotopy invariant (Proposition 2, section 5). This allows us to introduce the notion of a pair (M, \mathcal{W}) to be of determinant class. We point out that for $\mathcal{A} = \mathbb{R}$ any pair (M, \mathcal{W}) is of determinant class.

Theorem 2. *Let M be a closed manifold of odd dimension d and \mathcal{W} an $(\mathcal{A}, \Gamma^{op})$ -Hilbert module of finite type. If the pair (M, \mathcal{W}) is of determinant class then, for any Riemannian metric g and any generalized triangulation τ of M , $\log T_{\text{an}}(M, g, \mathcal{W})$ and $\log T_{\text{Re}}(M, \tau, g, \mathcal{W})$ are both finite, real numbers and*

$$\log T_{\text{an}} = \log T_{\text{Re}}.$$

Let us make a few comments concerning Theorem 2. First note that if M is of even dimension, then Theorem 2 is also true. In this case, however, it is an immediate consequence of the Poincaré duality, which implies that both torsions are equal to 1.

If $\mathcal{A} = \mathbb{R}$, the $(\mathbb{R}, \Gamma^{op})$ -Hilbert modules of finite type are precisely the orthogonal Γ -representations and Theorem 2 reduces to the Cheeger-Müller Theorem ([Ch],[Mü]) and, when specialized to this situation, we thus obtain a new proof of their theorem.

It suffices to prove Theorem 2 for \mathcal{W} a free \mathcal{A} -module. This follows easily from the following three facts:

- 1): If $\mathcal{A} = \mathbb{R}$ then \mathcal{W} is \mathbb{R} free and the pair (M, \mathcal{W}) is of determinant class.
- 2): If \mathcal{W} is Γ^{op} -trivial then (M, \mathcal{W}) is of determinant class, $\log T_{\text{an}}(M, g, \mathcal{W}) = \dim_N \mathcal{W} \cdot \log T_{\text{an}}(M, g)$ and $\log T_{\text{Re}}(M, \tau, g, \mathcal{W}) = \dim_N \mathcal{W} \cdot \log T_{\text{Re}}(M, \tau, g)$, where $\log T_{\text{an}}(M, g)$ and $\log T_{\text{Re}}(M, \tau, g)$ denote the analytic and Reidemeister torsions of the Riemannian manifold (M, g) and the trivial 1-dimensional representation of $\pi_1(M)$.
- 3): Suppose that \mathcal{W}_1 and \mathcal{W}_2 are two $(\mathcal{A}, \Gamma^{op})$ -Hilbert modules of finite type and $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$. If (M, \mathcal{W}) and (M, \mathcal{W}_2) are of determinant class then (M, \mathcal{W}_1) is of determinant class and the analytic resp. Reidemeister torsion of (M, g, \mathcal{W}) is the product of the analytic resp. Reidemeister torsion of (M, g, \mathcal{W}_1) with the analytic resp. Reidemeister torsion (M, g, \mathcal{W}_2) .

In the case when $\mathcal{A} = \mathcal{N}(\Gamma)$, the von Neumann algebra of bounded operators associated to Γ , acting on $\mathcal{W} = l_2(\Gamma)$, and \mathcal{W} is viewed as a $(\mathcal{N}(\Gamma), \Gamma^{op})$ -Hilbert module of finite type, then Theorem 2 reduces to Theorem 1.

Theorem 2 is derived from Corollary C (section 6.2), a relative version of Theorem 2, using product formulas for the analytic torsion and the Reidemeister torsion (section 4) and the metric anomaly (Lemma 6.10). To state Corollary C let M and M' be two closed manifolds of the same dimension with fundamental groups isomorphic to Γ , and assume that they are equipped with generalized triangulations $\tau = (g, h)$ and $\tau' = (g', h')$ such that the functions h and h' have the same number of critical points for each index. Then, for an arbitrary $(\mathcal{A}, \Gamma^{op})$ -Hilbert module of finite type \mathcal{W} , with (M, \mathcal{W}) and (M', \mathcal{W}') of determinant class

$$\log T_{\text{an}} - \log T'_{\text{an}} = \log T_{\text{Re}} - \log T'_{\text{Re}}.$$

An important ingredient in the proof of Corollary C is the Witten deformation of the de Rham complex associated with a generalized triangulation $\tau = (g, h)$. In section 5 we extend the analysis of Helffer-Sjöstrand [HS1] of the Witten complex to the analogous complex constructed for differential forms on M with coefficients in \mathcal{W} . Although the results and the estimates in the case where \mathcal{A} is an arbitrary finite von Neumann algebra are similar to those obtained by Helffer-Sjöstrand in the case $\mathcal{A} = \mathbb{R}$, additional arguments are necessary since the spectrum of the Laplacians $\Delta_q(t)$ is typically not discrete.

The Witten deformation permits us to define smooth functions $\log T_{\text{an}}(h, t)$, $\log T_{\text{sm}}(h, t)$ and $\log T_{\text{la}}(h, t)$ with $\log T_{\text{an}}(h, 0) = \log T_{\text{an}}$ where $\log T_{\text{an}}(h, t) = \log T_{\text{sm}}(h, t) + \log T_{\text{la}}(h, t)$ is a decomposition of $\log T_{\text{an}}(h, t)$ into a part $\log T_{\text{sm}}(h, t)$ which corresponds to the small spectrum of the Laplacians $\Delta_q(t)$ and a complimentary part $\log T_{\text{la}}(h, t)$. The results presented in sections 2 and 3 lead to the conclusion that these three functions have asymptotic expansion when $t \rightarrow \infty$. The free term of such an expansion refers to the 0'th order coefficient of the expansion as $t \rightarrow \infty$.

The results presented in sections 3 and 5 permit us to show that the free term of

$$\log T_{\text{an}}(h, t) - \log T_{\text{sm}}(h, t) - \log T_{\text{an}}(h', t) - \log T_{\text{sm}}(h', t)$$

is equal to

$$\log T_{\text{an}} - \log T_{\text{Re}} - \log T'_{\text{an}} - \log T'_{\text{Re}}.$$

Using the Mayer Vietoris type formula (cf section 3) we show that the free term of $\log T_{\text{la}}(h, t) - \log T_{\text{la}}(h', t)$ is equal to zero and we can therefore conclude Corollary C.

The paper is organized as follows:

In section 1 we recall, for the convenience of the reader, the concepts of a finite von Neumann algebra \mathcal{A} , an \mathcal{A} -Hilbert module of finite type, a finite (von Neumann) dimensional representation of a group, determinants in the von Neumann sense and the torsion of a finite complex of \mathcal{A} -Hilbert modules of finite type. This section is entirely expository.

In section 2 and 3 we describe the theory of pseudodifferential operators acting on sections of a given bundle $\mathcal{E} \rightarrow M$ of \mathcal{A} -Hilbert modules of finite type. In particular,

we extend Seeley's result on zeta-functions for elliptic pseudodifferential operators and the corresponding regularized determinants, as well as the results of [BFK2], to the extent needed in this paper, for this new class of operators. The calculus of such operators is not new, but we failed to find a reference suited for our needs (cf e.g. [FM],[Le] and [Lu] for related work).

In section 4 we define the analytic torsion and Reidemeister torsion and we prove a product formula for each of them. These product formulas are slight generalizations of the product formulas presented in [L] and [CM], but for the convenience of the reader we include the proofs.

In section 5, we discuss the Witten deformation of the deRham complex of M with coefficients in a representation of finite von Neumann dimension. Moreover using the Witten deformation we prove Proposition 2.

In section 6 we present the proof of Theorem 2.

One can generalize the analytic and Reidemeister torsions associated to (M, g, \mathcal{W}) to include additional data, for example a finite dimensional hermitian vector bundle on M equipped with a flat connection . By the same methods as presented in this paper one can prove a result which compares these two generalized torsions. In the case $\mathcal{A} = \mathbb{R}$ such type of result was first established in ([BZ]), compare also ([BFK1].

Using the same arguments as in ([Lü1]), one can extend Theorem 2 to compact manifolds with boundary. Both extensions are useful for the calculations of the L_2 torsions; together with some applications they will be presented in a forthcoming paper.

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1. LINEAR ALGEBRA IN THE VON NEUMANN SENSE

In this section we collect for the convenience of the reader a number of results concerning linear algebra in the von Neumann sense (cf e.g. [CM],[Co],[Di],[GS],[LR] for reference).

1.1 \mathcal{A} -Hilbert modules.

Definition 1.1. A finite von Neumann algebra \mathcal{A} is a unital \mathbb{R} -algebra with a $*$ -operation and a trace $\text{tr}_N : \mathcal{A} \rightarrow \mathbb{R}$ which satisfies the following properties:

(T1) $\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$, defined by $\langle a, b \rangle = \text{tr}_N(ab^*)$, is a scalar product and the completion \mathcal{A}_2 of \mathcal{A} with respect to this scalar product is a separable Hilbert space.

(T2) \mathcal{A} is separable and weakly closed, when viewed as a subalgebra of $\mathcal{L}(\mathcal{A}_2) := \mathcal{L}(\mathcal{A}_2, \mathcal{A}_2)$, the space of linear, bounded operators on \mathcal{A}_2 , where elements of \mathcal{A} are identified with the corresponding left translations in $\mathcal{L}(\mathcal{A}_2)$ (a sequence $\{a_n\}_{n \geq 1}$ in \mathcal{A} converges weakly to $a \in \mathcal{A}_2$ if $\lim_{n \rightarrow \infty} \langle a_n x, y \rangle = \langle ax, y \rangle$ for all $x, y \in \mathcal{A}_2$).

(T3) The trace is normal, i.e for any monotone increasing net, $(a_i)_{i \in I}$, such that $a_i \geq 0$ and $a = \sup_{i \in I} a_i$ exists in \mathcal{A} , one has $\text{tr}_N a = \sup_{i \in I} \text{tr}_N a_i$. Here $a_i \geq 0$ means that $a_i = a_i^*$ and $\langle a_i x, x \rangle \geq 0$ for all $x \in \mathcal{A}$.

In the sequel, \mathcal{A} will always denote a finite von Neumann algebra. Introduce the opposite algebra \mathcal{A}^{op} of \mathcal{A} , where \mathcal{A}^{op} has the same underlying vector space, $|\mathcal{A}^{op}| = |\mathcal{A}|$, $*$ -operation, trace and unit element as \mathcal{A} , but the multiplication " \cdot_{op} " of the elements $a, b \in |\mathcal{A}^{op}|$ is defined by $a \cdot_{op} b = b \cdot a$. Note that \mathcal{A}^{op} is a finite von Neumann algebra as well. The right translation by elements of \mathcal{A} induces an embedding $r : \mathcal{A}^{op} \rightarrow \mathcal{L}(\mathcal{A}_2)$ which identifies \mathcal{A}^{op} with the subalgebra $\mathcal{L}_{\mathcal{A}}(\mathcal{A}_2) \subset \mathcal{L}(\mathcal{A}_2)$ of bounded \mathcal{A} -linear maps (with respect to the \mathcal{A} -module structure of \mathcal{A}_2 induced by left multiplication). Therefore we can introduce a trace on $\mathcal{L}_{\mathcal{A}}(\mathcal{A}_2)$, also denoted by tr_N , defined for $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{A}_2)$ by

$$\text{tr}_N(f) := \text{tr}_N(r^{-1}(f)).$$

Definition 1.2. (1) \mathcal{W} is an \mathcal{A} -Hilbert module if

(HM1) \mathcal{W} is a Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle$.

(HM2) \mathcal{W} is a left \mathcal{A} -module so that $\langle a^*v, w \rangle = \langle v, aw \rangle$ ($a \in \mathcal{A}; v, w \in \mathcal{W}$).

(HM3) \mathcal{W} is isometric to a closed submodule of $\mathcal{A}_2 \otimes V$ where V is a separable Hilbert space and the tensor product $\mathcal{A}_2 \otimes V$ is taken in the category of Hilbert spaces.

(2) \mathcal{W} is an \mathcal{A} -Hilbert module of finite type if \mathcal{W} is an \mathcal{A} -Hilbert module and

(HM4) \mathcal{W} is isometric to a closed submodule $\mathcal{A}_2 \otimes V$ where V is a finite dimensional vector space.

(3) A morphism $f : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ between \mathcal{A} -Hilbert modules of finite type, \mathcal{W}_1 and \mathcal{W}_2 , is a bounded, \mathcal{A} -linear operator; f is an isomorphism if it is bijective and both f and f^{-1} are morphisms.

Let \mathcal{W} be an \mathcal{A} -Hilbert module and v an element in \mathcal{W} . The map $i_v : \mathcal{A}_2 \rightarrow \mathcal{W}$, defined by $i_v(a) = av$, extends to an \mathcal{A} -linear bounded map $\mathcal{A}_2 \rightarrow \mathcal{W}$.

Definition 1.3. A collection $\{e_1, \dots, e_l\}, l \leq \infty$, of elements of \mathcal{W} is called a base of \mathcal{W} if

$$(1.1) \quad i : \oplus_{1 \leq \nu \leq l} (\mathcal{A}_2)_\nu \rightarrow \mathcal{W}$$

is an isomorphism where each $(\mathcal{A}_2)_\nu$ is a copy of \mathcal{A}_2 and $i = \sum_{1 \leq \nu \leq l} i_{e_\nu}$. The base is called orthonormal if, in addition, i is an isometry. A Hilbert module is free if it has a base.

If $l = \infty$ the direct sum in (1.1) is a Hilbert direct sum in the category of Hilbert spaces. Note that $\{e_1, \dots, e_l\}$ is an orthonormal basis iff the closed invariant subspace spanned by e_1, \dots, e_l is \mathcal{W} , and for any i, j and $a, b \in \mathcal{A}$, $\langle ae_i, be_j \rangle = \langle a, b \rangle \delta_{ij}$. Further if $\{e_1, \dots, e_l\}$ is a base then $\{f_1, \dots, f_l\}$ with $f_\nu = i(i^*i)^{-\frac{1}{2}}(0, \dots, 0, 1, 0, \dots, 0)$ is an orthonormal base of \mathcal{W} .

Proposition 1.4.

(1) Any \mathcal{A} -Hilbert module \mathcal{W} can be decomposed as $\mathcal{W} = \oplus_{1 \leq \nu \leq l} \mathcal{W}_\nu$ with $l \leq \infty$ and \mathcal{W}_ν a closed invariant subspace of \mathcal{A}_2 . Further \mathcal{W} is of finite type iff $l < \infty$.

(2) If \mathcal{W}' is a closed invariant subspace of \mathcal{W} then $\mathcal{W}' \simeq \oplus_{1 \leq \nu \leq l} \mathcal{W}'_\nu$ where \mathcal{W}'_ν is a closed invariant subspace of \mathcal{W}_ν .

Proof: Note that (1) follows from (2). To prove (2), denote by π_ν the projection of \mathcal{W} on \mathcal{W}_ν and consider the filtration of \mathcal{W}' ,

$$\mathcal{W}'(0) := \mathcal{W}' \supset \mathcal{W}'(1) \supset \mathcal{W}'(2) \supset \dots$$

where $\mathcal{W}'(\nu)$ is defined inductively $\mathcal{W}'(\nu) = \text{Null}(\pi_\nu) \cap \mathcal{W}'(\nu - 1)$. It follows that $\mathcal{W}' \simeq \oplus_{1 \leq \nu < l} \mathcal{W}'(\nu) / \mathcal{W}'(\nu + 1) \subset \mathcal{W}_\nu$. \diamond

Let \mathcal{W} be an \mathcal{A} -Hilbert module of finite type. The algebra $\mathcal{L}_\mathcal{A}(\mathcal{W}) := \mathcal{L}_\mathcal{A}(\mathcal{W}, \mathcal{W})$ of bounded \mathcal{A} -linear operators on \mathcal{W} is a finite von Neumann algebra, whose trace is defined in the following way. First assume that the module \mathcal{W} is free. Choose a basis $\{e_1, \dots, e_l\}$. With respect to this basis an operator $A \in \mathcal{L}_\mathcal{A}(\mathcal{W})$ has a matrix representation $(a_{ij})_{1 \leq i, j \leq l}$, $i, j = 1, \dots, l$, with entries a_{ij} in $\mathcal{L}_\mathcal{A}(\mathcal{A}_2) = \mathcal{A}^{op}$. Define $\text{tr}_N(A) = \sum_{i=1}^l a_{ii}$. One shows that $\text{tr}_N(A)$ is independent of the chosen basis and therefore well defined. In the general case \mathcal{W} is a closed invariant subspace of a free \mathcal{A} -Hilbert module \mathcal{V} of finite type. We write $\mathcal{V} = \mathcal{W} \oplus \mathcal{W}^\perp$ and consider $\tilde{A} = A \oplus 0 \in \mathcal{L}_\mathcal{A}(\mathcal{V}, \mathcal{V})$. Define $\text{tr}_N(A) = \text{tr}_N(\tilde{A})$. One shows that $\text{tr}_N(A)$ is independent of the choice of \mathcal{V} . \diamond

For an \mathcal{A} -Hilbert module \mathcal{W} of finite type one defines the dimension $\dim_N(\mathcal{W})$ in the von Neumann sense by $\dim_N \mathcal{W} := \text{tr}_N \text{Id}_\mathcal{W}$. If \mathcal{W} is not of finite type one sets $\dim_N \mathcal{W} := \sup\{\dim_N \mathcal{W}'; \mathcal{W}' \text{ closed submodule of finite type}\}$. The von Neumann dimension is always a nonnegative real number or $+\infty$.

Remark 1.5 If \mathcal{W}_1 and \mathcal{W}_2 are \mathcal{A} -Hilbert modules, such that \mathcal{W}_1 is a closed invariant subspace of \mathcal{W}_2 and $\dim_N(\mathcal{W}_1) = \dim_N(\mathcal{W}_2) < \infty$, then $\mathcal{W}_1 = \mathcal{W}_2$. The von Neumann dimension of a Hilbert direct sum is the sum (possibly infinite) of the von Neumann dimension of the summands.

The following proposition is well known.

Proposition 1.6. Assume that \mathcal{W}_1 and \mathcal{W}_2 are \mathcal{A} -Hilbert modules of finite type.

- (1) If $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ and $g \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_2, \mathcal{W}_1)$ then $\text{tr}_N(fg)^n = \text{tr}_N(gf)^n$ for any $n \geq 1$.
- (2) If $f : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ is an isomorphism and $\alpha_i \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_i)$, $i = 1, 2$ so that $f \cdot \alpha_1 = \alpha_2 \cdot f$ then $\text{tr}_N \alpha_1 = \text{tr}_N \alpha_2$.

If \mathcal{A}' and \mathcal{A}'' are two finite von Neumann algebras the tensor product $\mathcal{A}' \otimes \mathcal{A}''$ is defined as the weak closure of the image of the algebraic tensor product of \mathcal{A}' and \mathcal{A}'' in $\mathcal{L}(\mathcal{A}'_2 \otimes \mathcal{A}''_2)$. The algebra $\mathcal{A}' \otimes \mathcal{A}''$ is again a finite von Neumann algebra whose trace has the property that $\text{tr}_N(a' \otimes a'') = \text{tr}_N a' \text{tr}_N a''$. If \mathcal{W}' and \mathcal{W}'' are \mathcal{A}' - resp. \mathcal{A}'' -Hilbert modules of finite type then $\mathcal{W}' \otimes \mathcal{W}''$ is an $\mathcal{A}' \otimes \mathcal{A}''$ -Hilbert module of finite type; moreover, given $f' \in \mathcal{L}_{\mathcal{A}'}(\mathcal{W}')$ and $f'' \in \mathcal{L}_{\mathcal{A}''}(\mathcal{W}'')$, $\text{tr}_N(f' \otimes f'') = \text{tr}_N f' \text{tr}_N f''$.

Definition 1.7. A morphism $f : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ is a weak isomorphism iff $\text{Null}(f) = 0$ and $\overline{\text{Range}(f)} = \mathcal{W}_2$.

Using polar decomposition a weak isomorphism $f : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ can be factored as $f = gf'$ where $f' : \mathcal{W}_1 \rightarrow \mathcal{W}_1$ is a weak isomorphism and $g : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ is an isometric isomorphism given by $f' = (f^* f)^{1/2}$, $g = f \cdot (f^* f)^{-1/2}$.

1.2 Determinant in the von Neumann sense.

Throughout this subsection we consider only \mathcal{A} -Hilbert modules of finite type.

Definition 1.8.

- (1) π is an Agmon angle for $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$ iff there exists $\epsilon > 0$ so that $\text{spec}(f) \cap V_{\pi, \epsilon} = \emptyset$ with $V_{\pi, \epsilon}$ defined as in the introduction.
- (2) π is a weak Agmon angle for $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$ iff $\text{spec}(f) \cap V_{\pi, 0} = \emptyset$

We will first treat the case where π is an Agmon angle. In particular this implies that f is an isomorphism. Define the complex powers of f , $f^s \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$, $s \in \mathbb{C}$, by the formula

$$(1.2) \quad f^s = \frac{1}{2\pi i} \int_{\gamma} \lambda^s (\lambda - f)^{-1} d\lambda,$$

where λ^s is a branch of the complex power s defined on $\mathbb{C}_{\pi} = \mathbb{C} \setminus \{z = \rho e^{i\pi}; \rho \in [0, \infty)\}$ and γ is a closed contour in \mathbb{C}_{π} which surrounds the compact set $\text{spec} f$ in \mathbb{C}_{π} with counterclockwise orientation. By Cauchy's theorem the contour γ in (1.2) can be replaced by the contour $\gamma_{\pi, \epsilon} = \gamma_1 \cup \gamma_2 \cup \gamma_3$ where $\gamma_1 := \{z = \rho e^{i\pi}; \infty \leq \rho \leq \epsilon/2\}$, $\gamma_2 := \{z = \frac{\epsilon}{2} e^{i\alpha}; \pi \geq \alpha \geq -\pi\}$ and $\gamma_3 := \{z = \rho e^{i(-\pi)}; \frac{\epsilon}{2} \leq \rho \leq \infty\}$. Notice that f^s is an entire function in $s \in \mathbb{C}$ with values in $\mathcal{L}_{\mathcal{A}}(\mathcal{W})$ and $\text{tr}_N(f^s)$ is an entire function on \mathbb{C} . Therefore, if π is an Agmon angle for f , we can define the determinant $\det_N f$ in the von Neumann sense by

$$(1.3) \quad \log \det_N f = \frac{d}{ds} \Big|_{s=0} \text{tr}_N(f^s).$$

We remark that this notion of determinant is equivalent to the definition introduced by Fuglede and Kadison [FK].

If $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ is an isomorphism then f^*f is a selfadjoint positive isomorphism and one shows that $\det_N(f^*f) > 0$. Define

$$\text{Vol}_N f := (\det_N(f^*f))^{1/2}.$$

Proposition 1.9. (1) Suppose $f_t \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$, with t in an interval $I \subset \mathbb{R}$, is a family of class C^1 of morphisms and π is an Agmon angle for all of them. Then $\log \det_N(f_t)$ is of class C^1 and

$$(1.4) \quad \frac{d}{dt} \log \det_N(f_t) = \text{tr}_N\left(\left(\frac{d}{dt} f_t\right) f_t^{-1}\right).$$

(2) Suppose $f_i \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_i)$, $i = 1, 2$ with \mathcal{W}_1 and \mathcal{W}_2 \mathcal{A} -Hilbert modules of finite type and $\alpha : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ is an isomorphism so that $\alpha f_1 = f_2 \alpha$. Then the following statements hold:

(a) $\text{spec} f_1 = \text{spec} f_2$ and therefore π is an Agmon angle for f_1 iff it is an Agmon angle for f_2 . In this case $\log \det_N f_1 = \log \det_N f_2$.

(b) f_1 is an isomorphism iff f_2 is an isomorphism. In this case $\text{Vol}_N f_1 = \text{Vol}_N f_2$.

(3) Suppose $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1 \oplus \mathcal{W}_2)$ is of the form

$$f = \begin{pmatrix} f_1 & 0 \\ g & f_2 \end{pmatrix}.$$

Then the following statements hold:

(a) $\text{spec} f = \text{spec} f_1 \cup \text{spec} f_2$ and therefore π is an Agmon angle for f iff it is an Agmon angle for both f_1 and f_2 . In this case

$$(1.5A) \quad \log \det_N f = \log \det_N f_1 + \log \det_N f_2.$$

(b) f is an isomorphism iff f_1 and f_2 are both isomorphisms. In this case

$$(1.5B) \quad \log \text{Vol}_N f = \log \text{Vol}_N f_1 + \log \text{Vol}_N f_2.$$

(4) Suppose \mathcal{W}_1 , \mathcal{W}_2 and \mathcal{W}_3 are \mathcal{A} -Hilbert modules of finite type. If $f_1 \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ and $f_2 \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_2, \mathcal{W}_3)$ are isomorphisms then $f_2 \cdot f_1 \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_3)$ is an isomorphism and

$$(1.6) \quad \log \text{Vol}_N(f_2 \cdot f_1) = \log \text{Vol}_N f_1 + \log \text{Vol}_N f_2.$$

(5) If $\alpha_i \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_i)$, $i = 1, 2$ are isometries and $f : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ is an isomorphism then $\alpha_2 f \alpha_1 \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ is an isomorphism and

$$\log \text{Vol}_N(\alpha_2 f \alpha_1) = \log \text{Vol}_N f.$$

Proof All the statements can be proved in an elementary way. For the convenience of the reader we indicate how to prove formula (1.6). We have to prove that

$$\log \det_N f_1^* f_2^* f_2 f_1 - \log \det_N f_1^* f_1 = \log \det_N f_2^* f_2.$$

Consider the 1-parameter family $C(t)$ of positive, selfadjoint operators, $C(t) := f_1^* (f_2^* f_2)^t f_1$, defined on \mathcal{W}_1 . Using formula (1.4) one verifies that

$$\frac{d}{dt} \log \det_N f_1^* (f_2^* f_2)^t f_1 = \frac{d}{dt} \log \det_N (f_2^* f_2)^t.$$

This leads to the claimed formula,

$$\begin{aligned} \log \det_N f_1^* f_2^* f_2 f_1 - \log \det_N f_1^* f_1 &= \\ \int_0^1 \frac{d}{dt} \log \det_N (f_1^* (f_2^* f_2)^t f_1) dt &= \\ \int_0^1 \frac{d}{dt} \log \det_N (f_2^* f_2)^t dt &= \log \det_N (f_2^* f_2). \quad \diamond \end{aligned}$$

First let us treat the case where π is a weak Agmon angle for $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$. In this case π is an Agmon angle for $f + \lambda$ with any $\lambda > 0$. One verifies that $\log \det_N (f + \lambda)$ is a real analytic function in $\lambda \in (0, \infty)$. We define $\log \det_N f$ as the element in \mathbf{D} (cf. Introduction), represented by the real analytic function

$$(1.8) \quad \log \det_N (f + \lambda) - \log \lambda \dim_N \text{Null}(f).$$

We note that parts (2) and (3) of Proposition 1.9 extend to this case as well.

Let $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ be a weak isomorphism. For $\lambda \geq 0$ denote by $\mathcal{P}_f(\lambda)$ the set of all \mathcal{A} -invariant closed subspaces $\mathcal{L} \subset \mathcal{W}_1$ such that, for $x \in \mathcal{L}$, $\|f(x)\| \leq \sqrt{\lambda} \|x\|$. Following Gromov-Shubin ([GS]) introduce the function $F_f : [0, \infty) \rightarrow [0, \infty)$ defined by $F_f(\lambda) := \sup\{\dim_N \mathcal{L}; \mathcal{L} \in \mathcal{P}_f(\lambda)\}$. Observe that the function $F_f(\lambda)$ is nondecreasing, left continuous, $F_f(0) = 0$ and $F_f(\lambda) = \dim_N(\mathcal{W})$ for $\lambda \geq \|f\|$. Note that f is an isomorphism iff there exists $\lambda_0 > 0$ s.t. $F_f(\lambda) = 0$ for $\lambda < \lambda_0$. The Novikov-Shubin invariant $\alpha(f)$ associated to a weak isomorphism $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ is defined by

$$(1.10) \quad \alpha(f) := \liminf_{\lambda \rightarrow 0} \frac{\log F_f(\lambda)}{\log \lambda} \in [0, \infty].$$

Note that $\alpha(f) = \infty$ if f is an isomorphism.

If $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ is an arbitrary morphism let

$$(1.11) \quad \bar{f} : \mathcal{W}'_1 = \mathcal{W}_1 / \text{Null}(f) \rightarrow \overline{\text{Range } f} = \mathcal{W}'_2.$$

Note that \bar{f} is a weak isomorphism and define $\alpha(f)$ by

$$(1.12) \quad \alpha(f) := \alpha(\bar{f}).$$

Proposition 1.10. (1) For any weak isomorphism $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$

$$F_f(\lambda) = F_{(f^*f)^{1/2}}(\lambda) = F_{f^*}(\lambda).$$

(2) If $f : \mathcal{W}_1 \oplus \mathcal{W}_2 \rightarrow \mathcal{W}_1 \oplus \mathcal{W}_2$ is a weak isomorphism of the form

$$f = \begin{pmatrix} f_1 & 0 \\ g & f_2 \end{pmatrix},$$

then f_1 and f_2 are both weak isomorphisms and

$$\sup\{F_{f_1}(\lambda), F_{f_2}(\lambda)\} \leq F_f(\lambda) \leq F_{f_1}(\lambda) + F_{f_2}(\lambda).$$

(3) If $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$ is nonnegative and selfadjoint, define the spectral projectors $Q_f(\lambda) \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$ corresponding to the interval $[0, \lambda]$ and $N_f(\lambda) := \text{tr}_N Q_f(\lambda)$. For $\lambda \geq 0$, and \bar{f} given by (1.11)

$$(1.13) \quad N_f(\lambda) = \dim_N(\text{Null}(f)) + F_{\bar{f}}(\lambda).$$

The function $N_f(\lambda)$ is called the spectral function of f . Note that $F_{\bar{f}}(\lambda)$ is nondecreasing and $F_{\bar{f}}(0) = 0$. $F_{\bar{f}}(\lambda)$ can be used to represent $\log \det_N f$ as a Stieltjes integral $\int_{0+}^{\infty} \log(\mu + \lambda) dF_{\bar{f}}(\mu)$.

Denote by \mathbf{F} the set of functions $F : [0, \infty) \rightarrow [0, \infty)$ satisfying

- (1) $F(0) = 0$;
- (2) $F(\lambda)$ is nondecreasing;
- (3) F is continuous to the left,

and recall the following definitions of Gromov-Shubin (cf [GS])

Definition 1.11. (1) Functions $F, G \in \mathbf{F}$ are said to be dilational equivalent, denoted $F \stackrel{d}{\sim} G$, iff there exists $C > 0$ such that for $\lambda \geq 0$

$$(1.14) \quad G(C^{-1}\lambda) \leq F(\lambda) \leq G(C\lambda)$$

(2) Functions $F, G \in \mathbf{F}$ are said to be dilational equivalent near zero, denoted $F \stackrel{d}{\sim}_0 G$ iff there exist $C > 0$ and $\lambda_0 > 0$ such that (1.14) holds for $\lambda < \lambda_0$.

We end this subsection with the following observation. Suppose that $\psi : \mathcal{A}' \rightarrow \mathcal{A}''$ is a homomorphism of finite von Neumann algebras which preserves the units and the traces. Then ψ is injective. In particular, if \mathcal{A}'' is an \mathcal{A}' -Hilbert module of finite type, then any \mathcal{A}'' -Hilbert module of finite type \mathcal{W} is an \mathcal{A}' -Hilbert module of finite type and $\mathcal{L}_{\mathcal{A}''}(\mathcal{W}) \subseteq \mathcal{L}_{\mathcal{A}'}(\mathcal{W})$.

Remark 1.12 Assume that $\dim_{N, \mathcal{A}'} \mathcal{A}'' = r \in \mathbb{R}$.

- (1) $\dim_{N, \mathcal{A}'}(\mathcal{W}) = r \dim_{N, \mathcal{A}''}(\mathcal{W})$.
- (2) If $f \in \mathcal{L}_{\mathcal{A}''}(\mathcal{W})$ then $\text{tr}_{N, \mathcal{A}'}(f) = r \text{tr}_{N, \mathcal{A}''}(f)$.
- (3) If π is a weak Agmon angle for f then

$$\log \det_{N, \mathcal{A}'}(f) = r \log \det_{N, \mathcal{A}''}(f).$$

1.3 Cochain complexes of finite type and torsion in the von Neumann sense.

Definition 1.13. A cochain complex in the category of \mathcal{A} -Hilbert modules of finite type, $\mathcal{C} = (\mathcal{C}_i, d_i)$, consists of a collection of Hilbert modules of finite type \mathcal{C}_i , all but finitely many zero, and a collection of morphisms $d_i : \mathcal{C}_i \rightarrow \mathcal{C}_{i+1}$ which satisfies $d_i d_{i-1} = 0$. In the sequel we always assume that $\mathcal{C}_i = 0$ for $i < 0$ and refer to such a complex as a cochain complex of finite type over \mathcal{A} , or simply as a cochain complex of finite type.

The reduced cohomology of \mathcal{C} , $\overline{H}^i(\mathcal{C})$, is defined by

$$\overline{H}^i(\mathcal{C}) = \text{Null}(d_i) / \overline{\text{Range}(d_{i-1})}.$$

Define the Betti numbers and Euler-Poincaré characteristic of \mathcal{C} by

$$(1.15) \quad \beta_i(\mathcal{C}) := \dim_N \overline{H}^i(\mathcal{C}); \quad \chi(\mathcal{C}) := \sum_i (-1)^i \beta_i(\mathcal{C}),$$

and introduce a weighted version of the Euler-Poincaré characteristic,

$$(1.16) \quad \psi(\mathcal{C}) := \sum_i (-1)^i i \beta_i(\mathcal{C}).$$

Denote by $d_i^* : \mathcal{C}_{i+1} \rightarrow \mathcal{C}_i$ the adjoint of d_i , and consider $\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^*$. The operator Δ_i is a selfadjoint and nonnegative morphism.

Definition 1.14. (1) Given two cochain complexes of finite type over \mathcal{A} , \mathcal{C}' and \mathcal{C}'' , a morphism $\mathbf{f} : \mathcal{C}' \rightarrow \mathcal{C}''$ is given by a collection of morphisms $f_i : \mathcal{C}'_i \rightarrow \mathcal{C}''_i$ which commute with the differentials d_j .

(2) A homotopy \mathbf{t} between \mathbf{f} and \mathbf{g} is given by a collection of morphisms $t_i : \mathcal{C}'_i \rightarrow \mathcal{C}''_{i-1}$ satisfying

$$(1.17) \quad f_i - g_i = d_{i-1}'' t_i + t_{i+1} d_i'.$$

(3) Two cochain complexes \mathcal{C}' and \mathcal{C}'' are homotopy equivalent if there exist morphisms (in the category of complexes) $\mathbf{f} : \mathcal{C}' \rightarrow \mathcal{C}''$ and $\mathbf{g} : \mathcal{C}'' \rightarrow \mathcal{C}'$ so that $\mathbf{g} \cdot \mathbf{f}$ resp. $\mathbf{f} \cdot \mathbf{g}$ is homotopic to $\text{id}_{\mathcal{C}'}$ resp. $\text{id}_{\mathcal{C}''}$.

Given a morphism $\mathbf{f} : \mathcal{C}' \rightarrow \mathcal{C}''$ denote by $\overline{H}(\mathbf{f})^i$ the induced morphisms of \mathcal{A} -Hilbert modules $\overline{H}(\mathbf{f})^i : \overline{H}^i(\mathcal{C}') \rightarrow \overline{H}^i(\mathcal{C}'')$. Note that if $f_i : \mathcal{C}' \rightarrow \mathcal{C}''$, $i = 1, 2$ are two homotopic morphisms then $\overline{H}(\mathbf{f}_1)^i = \overline{H}(\mathbf{f}_2)^i$ for all i . Given a finite type cochain complex $\mathcal{C} = (\mathcal{C}_i, d_i)$ each \mathcal{C}_i can be decomposed as a direct sum of mutually orthogonal subspaces $\mathcal{C}_i = \mathcal{H}_i \oplus \mathcal{C}_i^+ \oplus \mathcal{C}_i^-$ with

$$(1.18) \quad \mathcal{H}_i = \text{Null} \Delta_i; \quad \mathcal{C}_i^+ = \overline{d_{i-1}(\mathcal{C}_{i-1})}, \quad \mathcal{C}_i^- = \overline{d_{i+1}^*(\mathcal{C}_{i+1})}.$$

This decomposition is called the Hodge decomposition . The map d_i can then be described by a 3×3 matrix of the form

$$(1.19) \quad d_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \underline{d}_i \\ 0 & 0 & 0 \end{pmatrix}.$$

where $\underline{d}_i : \mathcal{C}_i^- \rightarrow \mathcal{C}_{i+1}^+$ is a weak isomorphism and the combinatorial Laplacian $\Delta_i = d_{i-1}d_{i-1}^* + d_i^*d_i$ then takes the form of the diagonal matrix $\text{diag}(0, d_{i-1}d_{i-1}^*, d_i^*d_i)$.

Let $\mathbf{f} : \mathcal{C}^1 \rightarrow \mathcal{C}^2$ be a morphism. With respect to the Hodge decompositions of \mathcal{C}_i^1 and \mathcal{C}_i^2 , the morphism $f_i : \mathcal{C}_i^1 \rightarrow \mathcal{C}_i^2$ can be written as a 3×3 -matrix of the form

$$(1.20) \quad f_i = \begin{pmatrix} f_{i,11} & 0 & f_{i,13} \\ f_{i,21} & f_{i,22} & f_{i,23} \\ 0 & 0 & f_{i,33} \end{pmatrix}.$$

where $f_{i,11} \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}_i^1, \mathcal{H}_i^2)$, $f_{i,22} \in \mathcal{L}_{\mathcal{A}}(\mathcal{C}_i^{1,+}, \mathcal{C}_i^{2,+})$, $f_{i,33} \in \mathcal{L}_{\mathcal{A}}(\mathcal{C}_i^{1,-}, \mathcal{C}_i^{2,-})$ and $\underline{d}_i^2 \cdot f_{i,33} = f_{i+1,22} \cdot \underline{d}_i^1$.

Definition 1.15. A cochain complex \mathcal{C} is called perfect if, for any i , \underline{d}_i is an isomorphism.

Lemma 1.16. (1) Given a cochain complex $\mathcal{C} = (\mathcal{C}_i, d_i)$ one can find a modification \tilde{d}_i of d_i so that $\tilde{\mathcal{C}} = (\mathcal{C}_i, \tilde{d}_i)$ is perfect and has the same Hodge decomposition as \mathcal{C} .

(2) Given an isomorphism $\mathbf{f} : \mathcal{C}^1 \rightarrow \mathcal{C}^2$ of cochain complexes $\mathcal{C}^k = (\mathcal{C}_i^k, d_i^k)$, $k = 1, 2$ one can find modifications \tilde{d}_i^k of d_i^k so that

$$(1.21) \quad f_{i+1} \tilde{d}_i^1 = \tilde{d}_i^2 f_i$$

and the cochain complexes $\tilde{\mathcal{C}}^k = (\mathcal{C}_i^k, \tilde{d}_i^k)$, with $k = 1, 2$, are perfect and have the same Hodge decompositions as \mathcal{C}^k .

Proof Statement (1) follows by choosing \tilde{d}_i of the form

$$\tilde{d}_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tilde{\underline{d}}_i \\ 0 & 0 & 0 \end{pmatrix}.$$

where $\tilde{\underline{d}}_i$ is the isometry in the polar decomposition of \underline{d}_i given by $\tilde{\underline{d}}_i = \underline{d}_i(\underline{d}_i^* \underline{d}_i)^{-\frac{1}{2}}$.

(2) With respect to the Hodge decomposition of \mathcal{C}_i^1 and \mathcal{C}_i^2 define $\tilde{\underline{d}}_i^1$ as in (1) and choose $\tilde{\underline{d}}_i^2$ to be of the form

$$\tilde{\underline{d}}_i^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tilde{\underline{d}}_i^2 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\tilde{\underline{d}}_i^2 := f_{i+1,22} \cdot \tilde{\underline{d}}_i^1 \cdot f_{i,33}^{-1}$.

In section 6.1 we will need the following

Proposition 1.17. *Suppose $\mathcal{C}(t) = (\mathcal{C}_i(t), d_i(t))$ is a family of cochain complexes of finite type depending on a parameter $t \geq 0$, and $\mathbf{f}(t) : \mathcal{C}(t) \rightarrow \mathcal{C}$ is an isomorphism of cochain complexes for any t . Introduce $\log V(t) := \sum_{q=0}^d (-1)^q \log \text{Vol}_N \overline{H}(\mathbf{f}(t))^q$. Assume that $\mathcal{C}_i(t)$ and \mathcal{C}_i are free modules, that $e_{i,1}(t), \dots, e_{i,l_i}(t)$ is an orthonormal base for $\mathcal{C}_i(t)$, $e_{i,1}, \dots, e_{i,l_i}$ is an orthonormal base for \mathcal{C}_i and that $f_i(t)$, when expressed with respect to these bases, is an $l_i \times l_i$ -matrix with entries in \mathcal{A}^{op} of the form $Id + O(1/t)$.*

Then $\log V(t) = O(1/t)$.

Proof In view of Lemma 1.16 it suffices to prove the result for the case where $\mathcal{C}(t)$ and \mathcal{C} are all perfect. In view of (1.20)

$$(1.22) \quad \log \text{Vol}_N \overline{H}(\mathbf{f}(t))^i = \log \text{Vol}_N(f_{i,11}(t)) = \log \text{Vol}_N(\pi_i f_i(t) I_i(t))$$

where $I_i(t)$ denotes the inclusion of $\mathcal{H}_i(t)$ in \mathcal{C}_i and π_i the orthogonal projection of \mathcal{C}_i onto \mathcal{H}_i . Denote by $P_i(t)$ the orthogonal projections of $\mathcal{C}_i(t)$ onto $\mathcal{H}_i(t)$. We obtain

$$(1.23) \quad \begin{aligned} \log \text{Vol}_N(\pi_i f_i(t) I_i(t)) &= \frac{1}{2} \log \det_N(P_i(t) f_i^*(t) \pi_i f_i(t) I_i(t)) = \\ &= \frac{1}{2} \log \det_N(P_i(t) f_i^*(t) f_i(t) I_i(t) - P_i(t) f_i^*(t) (Id - \pi_i) f_i(t) I_i(t)). \end{aligned}$$

In view of the assumptions on $f_i(t)$, one has $P_i(t) f_i^*(t) f_i(t) I_i(t) = Id + O(1/t)$. Next we will show that

$$(1.24) \quad P_i(t) f_i^*(t) (Id - \pi_i) f_i(t) I_i(t) = O(1/t).$$

For this purpose note that the assumption concerning the asymptotics of $f_i(t)$ combined with the hypothesis that all complexes are perfect imply the existence of $\epsilon > 0$ and $t_0 > 0$ such that for $t \geq t_0$, $\text{spec} \Delta_i(t) \setminus \{0\}$ and $\text{spec} \Delta_i \setminus \{0\}$ are contained in $[2\epsilon, \infty)$. Therefore

$$P_i(t) = \frac{1}{2\pi i} \int_{S_\epsilon} (z - \Delta_i(t))^{-1} dz, \text{ and } \pi_i = \frac{1}{2\pi i} \int_{S_\epsilon} (z - \Delta_i)^{-1} dz,$$

where S_ϵ is the circle in the complex plane of radius ϵ , centered at the origin. Using the assumptions on $f_i(t)$, one concludes that $\Delta_i \cdot f_i(t) = f_i(t) \cdot \Delta_i(t) + O(1/t)$, and in view of (1.23) one obtains

$$(1.25) \quad (Id - \pi_i) f_i(t) = f_i(t) (Id - P_i(t)) + O(1/t).$$

As $(Id - P_i(t)) I_i(t) = 0$, estimate (1.25) implies (1.24); formulas (1.23) and (1.24) lead to

$$(1.26) \quad \log \text{Vol}_N(\overline{H}(\mathbf{f}(t))^i) = \frac{1}{2} \log \det_N(Id + O(1/t)) = O(1/t).$$

Given a cochain complex \mathcal{C} of finite type, introduce, following Gromov-Shubin ([GS], cf. also end of section 1.2), the functions $F_{\mathcal{C},i}(\lambda) \in \mathbf{F}$ defined by $F_{\mathcal{C},i}(\lambda) := F_{\underline{d}_i}(\lambda)$ and the numbers α_i defined by $\alpha_i := \alpha(\underline{d}_i)$. The following result is due to Gromov-Shubin ([GS])

Proposition 1.18. *Suppose $\mathbf{f} : \mathcal{C}' \rightarrow \mathcal{C}''$ and $\mathbf{g} : \mathcal{C}'' \rightarrow \mathcal{C}'$ are two morphisms of cochain complexes so that $\mathbf{id}_{\mathcal{C}'}$ is homotopic to $\mathbf{g}\mathbf{f}$ by a homotopy $\mathbf{t} = \{t_i\}$. Then $F_{\mathcal{C}',i}(\lambda) \leq F_{\mathcal{C}'',i}(4\|f_{i+1}\|^2\|g_i\|^2\lambda)$ for $\lambda \leq \frac{1}{4\|t_{i+1}\|^2}$.*

In particular if $\mathbf{f} : \mathcal{C}' \rightarrow \mathcal{C}''$ is an isomorphism then $F_{\mathcal{C}',i}(\lambda) \stackrel{d}{\sim} F_{\mathcal{C}'',i}(\lambda)$ and if \mathcal{C}' and \mathcal{C}'' are homotopy equivalent, then $F_{\mathcal{C}',i}(\lambda) \stackrel{d}{\sim} F_{\mathcal{C}'',i}(\lambda)$, and therefore $\alpha'_i = \alpha''_i$.

The torsion $\log T(\mathcal{C})$ is the element in \mathbf{D} defined by

$$\log T(\mathcal{C}) = \frac{1}{2} \sum_i (-1)^{i+1} i \log \det_N \Delta_i.$$

Remark 1.19 The spectral functions $N_i(\lambda) := N_{\Delta_i}(\lambda)$ satisfies

$$(1.28) \quad N_i(\lambda) = \beta_i + F_{i-1}(\lambda) + F_i(\lambda)$$

and therefore $\log T(\mathcal{C})$ can be represented by the real analytic function in λ

$$(1.29) \quad \frac{1}{2} \sum_i (-1)^i \int_{0+}^{\infty} \log(\mu + \lambda) dF_i(\mu).$$

Definition 1.20. *The cochain complex of finite type \mathcal{C} is of determinant class iff $\int_{0+}^1 \log(\lambda) dN_i(\lambda) > -\infty$ for all i .*

We point out that if \mathcal{C} is of determinant class, then $\log T(\mathcal{C})$ is in $\mathbb{R} \subset \mathbf{D}$, and a sufficient condition for \mathcal{C} to be of determinant class is that $\alpha_k > 0$ for $0 \leq k \leq d$.

It will be convenient for the proof of the product formula below to introduce for $\lambda > 0$ and $s \in \mathbb{C}$ with $\Re s > 0$,

$$(1.30) \quad \zeta_{\mathcal{C}}(\lambda, s) = \frac{1}{2} \sum_i (-1)^i i \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{(-t\lambda)} \text{tr}_N e^{-t\Delta_i} dt.$$

This function is real analytic in λ , complex analytic in s for $\Re s > 0$ and admits an analytic continuation to the entire s -plane so that $s = 0$ is a regular point. Using that

$$(1.31) \quad \zeta_{\mathcal{C}}(\lambda, s) = \frac{1}{2} \sum_i (-1)^i i \text{tr}_N ((\Delta_i + \lambda)^{-s}).$$

one sees that $\log T(\mathcal{C})$ is also represented by the real analytic function in λ given by

$$(1.32) \quad \frac{d}{ds} \Big|_{s=0} \zeta_{\mathcal{C}}(\lambda, s) - \psi(\mathcal{C}) \log \lambda$$

where $\psi(\mathcal{C})$ denotes the weighted Euler-Poincaré characteristic. This perturbation of the ζ -function was also studied in [Go].

Suppose that \mathcal{A}_i ($i = 1, 2$) are finite von Neuman algebras. Note that $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ (tensor product in the category of finite von Neumann algebras) is also a finite von Neumann algebra. If \mathcal{W}_i are \mathcal{A}_i -Hilbert modules of finite type then the tensor product $\mathcal{W}_1 \otimes \mathcal{W}_2$ (tensor product in the category of Hilbert spaces) is an \mathcal{A} -Hilbert module of finite type and

$$\dim_N(\mathcal{W}_1 \otimes \mathcal{W}_2) = \dim_N \mathcal{W}_1 \cdot \dim_N \mathcal{W}_2.$$

Let \mathcal{C}' resp. \mathcal{C}'' , be two cochain complexes of finite type over \mathcal{A}_1 , resp. \mathcal{A}_2 . Denote by $\mathcal{C} = \mathcal{C}' \otimes \mathcal{C}''$ the tensor product of these cochain complexes,

$$\mathcal{C}_i = \sum_{p+r=i} \mathcal{C}'_p \otimes \mathcal{C}''_r, \quad \sum_{p+r=i} d_i = d'_p \otimes id + (-1)^p id \otimes d''_r.$$

Then \mathcal{C} is a cochain complex of finite type over \mathcal{A} .

Proposition 1.21. (cf [CM], [LR]) *Let \mathcal{C}' , resp. \mathcal{C}'' be two finite type cochain complexes over \mathcal{A}_1 resp. \mathcal{A}_2 . Then, with $\mathcal{C} = \mathcal{C}' \otimes \mathcal{C}''$,*

(1)

$$\overline{H}^i(\mathcal{C}) = \sum_{p+r=i} \overline{H}^p(\mathcal{C}') \otimes \overline{H}^r(\mathcal{C}'')$$

(2)

$$\zeta_{\mathcal{C}}(\lambda, s) = \zeta_{\mathcal{C}'}(\lambda, s)\chi(\mathcal{C}'') + \zeta_{\mathcal{C}''}(\lambda, s)\chi(\mathcal{C}')$$

(3)

$$\psi(\mathcal{C}) = \psi(\mathcal{C}')\chi(\mathcal{C}'') + \psi(\mathcal{C}'')\chi(\mathcal{C}').$$

Proof: The proof of (1) can be found e.g. in [LR, Theorem 3.16] and (3) follows from (1). To prove (2) (cf [CM]) let \mathcal{C} be a cochain complex of finite type over \mathcal{A}_1 , and \mathcal{N} be a \mathcal{A}_2 -Hilbert module of finite type. We will show below that

$$(1.33) \quad \sum_q (-1)^q \text{tr}_N(e^{-t\Delta_q}) = \chi(\mathcal{C}).$$

Therefore if $\beta : \mathcal{N} \rightarrow \mathcal{N}$ is a morphism, then

$$(1.34) \quad \sum_q (-1)^q \text{tr}_N(e^{-t\Delta_q} \otimes \beta) = \text{tr}_N(\beta)\chi(\mathcal{C}).$$

To prove (1.33) we use the matrix representation of Δ_q with respect to the Hodge decomposition, $\text{diag}(0, \underline{d}_{q-1}^*, \underline{d}_q^*, \underline{d}_q)$ and Proposition 1.7 (1); they give

$$\text{tr}_N e^{-t\Delta_{q-1}}|_{\mathcal{C}_q^-} = \text{tr}_N e^{-t\Delta_q}|_{\mathcal{C}_q^+}$$

and consequently

$$\mathrm{tr}_N e^{-t\Delta_q} = \mathrm{tr}_N(e^{-t\Delta_q|_{c_q^+}}) + \mathrm{tr}_N(e^{-t\Delta_q|_{c_{q+1}^-}}) + \dim_N \mathrm{Null}(\Delta_q)$$

which leads to (1.33). Next, decompose $\Delta_q = \bigoplus_{p+r=q} \Delta_{p,r}$, where $\Delta_{p,r} = \Delta'_p \otimes id + id \otimes \Delta''_r$ to obtain

$$\begin{aligned} 2\zeta_{\mathcal{C}}(\lambda, s) &= \sum_{p,r} (-1)^{(p+r)} (p+r) \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda} \mathrm{tr}_N(e^{-t\Delta'_p} \otimes e^{-t\Delta''_r}) dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda} \mathrm{tr}_N(\{\bigoplus_p (-1)^p p e^{-t\Delta'_p}\} \otimes \{\bigoplus_r (-1)^r e^{-t\Delta''_r}\}) dt \\ &\quad + \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda} \mathrm{tr}_N(\{\bigoplus_p (-1)^p e^{-t\Delta'_p}\} \otimes \{\bigoplus_r (-1)^r r e^{-t\Delta''_r}\}) dt \end{aligned}$$

which, in view of (1.34), is equal to

$$\begin{aligned} &\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda} \left(\chi(\mathcal{C}'') \sum_p (-1)^p p \mathrm{tr}_N e^{-t\Delta'_p} + \chi(\mathcal{C}') \sum_r (-1)^r r \mathrm{tr}_N e^{-t\Delta''_r} \right) dt \\ &= 2\zeta_{\mathcal{C}'}(\lambda, s) \cdot \chi(\mathcal{C}'') + 2\zeta_{\mathcal{C}''}(\lambda, s) \cdot \chi(\mathcal{C}'). \end{aligned}$$

Corollary 1.22. (cf [CM]) *With the same assumptions as in Proposition 1.21, the following identity, viewed in the vector space \mathbf{D} , holds:*

$$\log T(\mathcal{C}) = \chi(\mathcal{C}'') \log T(\mathcal{C}') + \chi(\mathcal{C}') \log T(\mathcal{C}'').$$

1.4 $(\mathcal{A}, \Gamma^{op})$ -Hilbert module.

Definition 1.23. \mathcal{W} is an $(\mathcal{A}, \Gamma^{op})$ -Hilbert module of finite type if

(BM1) \mathcal{W} is an \mathcal{A} -Hilbert module of finite type;

(BM2) \mathcal{W} is a Γ^{op} -Hilbert module, defined by a unitary representation of Γ ;

(BM3) the action of \mathcal{A} and Γ^{op} commute.

Example Let X be a countable set. Denote by $l^2(X)$ the Hilbert space obtained by completion of $\mathbb{R}(X) = \{f : X \rightarrow \mathbb{R}; \mathrm{supp}(f) \text{ is finite}\}$ with respect to the scalar product

$$\langle f_1, f_2 \rangle := \sum_{x \in X} f(x)g(x).$$

Let Γ be a countable group and $\mathbb{R}(\Gamma)$ denote the unital \mathbb{R} -algebra with multiplication defined by convolution and $*$ -operation induced by the map $g \rightarrow g^{-1}$. The algebra $\mathbb{R}(\Gamma)$ has a finite trace given by $\mathrm{tr}(f) := f(e)$ where e denotes the unit element in Γ , and acts

from the left by convolutions on $l_2(\Gamma)$. This algebra can be viewed as a $*$ -subalgebra of $\mathcal{L}_\Gamma(l_2(\Gamma), l_2(\Gamma))$. Denote by $\mathcal{N}(\Gamma)$ its weak closure in $\mathcal{L}_\Gamma(l_2(\Gamma), l_2(\Gamma))$. Then $\mathcal{N}(\Gamma)$ is a finite von Neumann algebra.

Let $\rho : \Gamma \times X \rightarrow X$ be a *left* action of Γ on the set X with finite isotropy groups. ρ induces a left action of Γ by isometries which makes $l_2(X)$ an $\mathcal{N}(\Gamma)$ -Hilbert module ; if the quotient set $\Gamma \backslash X$ is finite, then this module is a Hilbert module of finite type. Suppose, in addition, that Γ' is another countable group and $\rho' : X \times \Gamma' \rightarrow X$ is a *right* action of Γ' on X so that ρ and ρ' commute. Γ' induces an action by isometries on $l_2(X)$ which makes $l_2(X)$ an $(\mathcal{N}(\Gamma), \Gamma'^{op})$ -Hilbert module of finite type. As an example, consider the case $X = |\Gamma|$, the underlying set of Γ , $\Gamma = \Gamma'$ and ρ and ρ' given by $\rho(g_1, g_2) = g_1 g_2$, and $\rho'(g_2, g_1) = g_2 g_1^{-1}$. Then $l_2(\Gamma)$ is an $(\mathcal{N}(\Gamma), \Gamma^{op})$ -Hilbert module of finite type, referred to as the *regular birepresentation*.

1.5 BUNDLES OF \mathcal{A} -HILBERT MODULES

Definition 1.24. A smooth bundle $p : \mathcal{E} \rightarrow M$ over a smooth manifold M is a bundle of \mathcal{A} -Hilbert modules of finite type with fiber \mathcal{W} if

(B1) $p : \mathcal{E} \rightarrow M$ is a smooth bundle of infinite dimensional topological vector spaces, equipped with a Hermitian structure μ which makes each fiber $p^{-1}(x), x \in M$, into a separable Hilbert space;

(B2) \mathcal{E} is equipped with a smooth fiberwise action $\rho : \mathcal{A} \times \mathcal{E} \rightarrow \mathcal{E}$ which makes each fiber $p^{-1}(x)$ an \mathcal{A} -Hilbert module of finite type.

(B3) \mathcal{W} is an \mathcal{A} -Hilbert module of finite type and $p : \mathcal{E} \rightarrow M$ is locally isomorphic to $p_o : \mathcal{W} \times M \rightarrow M$ where the local isomorphism intertwines p, p_o , the Hermitian structures and the \mathcal{A} -actions.

Example Let M be a closed smooth manifold with fundamental group $\Gamma := \pi_1(M)$ and let \mathcal{W} be an $(\mathcal{A}, \Gamma^{op})$ -Hilbert module of finite type. Let $\tilde{p} : \mathcal{W} \times \tilde{M} \rightarrow \tilde{M}$ be the trivial smooth bundle of \mathcal{A} -Hilbert modules; \tilde{p} is Γ -equivariant with respect to the diagonal action of Γ on $\mathcal{W} \times_\Gamma \tilde{M}$ and the left action of Γ on \tilde{M} . Therefore \tilde{p} induces $p : \mathcal{E} = \mathcal{W} \times_\Gamma \tilde{M} \rightarrow M$ which is a smooth bundle of \mathcal{A} -Hilbert modules of finite type. This bundle is the canonical bundle over M , associated to \mathcal{W} .

2. CALCULUS OF PSEUDODIFFERENTIAL OPERATORS ACTING ON \mathcal{A} -HILBERT BUNDLES OF FINITE TYPE

In this section we construct a calculus of pseudodifferential operators, called pseudodifferential \mathcal{A} -operators, on a compact manifold, where \mathcal{A} is a finite von Neumann algebra (cf e.g. [FM],[Le],[Lu] for related work).

2.1 Sobolev spaces, symbols and kernels.

Let B be a Banach space. For $u \in \mathcal{S}(\mathbb{R}^d, B)$, the space of functions $u : \mathbb{R}^d \longrightarrow B$ of Schwartz class, $\|u\|_s$ denotes the Sobolev s -norm given by

$$\|u\|_s^2 := \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \|\hat{u}(\xi)\|^2 d\xi$$

where $\hat{u}(\xi)$ denotes the Fourier transform of u .

Definition 2.1. (1) *The Sobolev space $H_s(\mathbb{R}^d, B)$ is the completion of $\mathcal{S}(\mathbb{R}^d, B)$ with respect to the Sobolev s -norm; equivalently, it can be defined as the space of all distributions $u \in \mathcal{S}'(\mathbb{R}^d, B)$ with*

$$(1 + |\xi|^2)^{s/2} \hat{u} \in L_2(\mathbb{R}^d; B).$$

(2) *The space $H_s^{loc}(\mathbb{R}, B)$ is the space of all distributions $u \in \mathcal{D}'(\mathbb{R}^d, B)$ such that $\phi u \in H_s(\mathbb{R}^d, B)$ for any $\phi \in C_0^\infty(\mathbb{R}^d)$.*

Note that the Sobolev spaces $H_s(\mathbb{R}^d, B)$ have the same properties as the usual Sobolev spaces except that Rellich's compactness theorem does not hold.

Let \mathcal{W} be an \mathcal{A} -Hilbert module. The space $H_s(\mathbb{R}^d, \mathcal{W})$ is an \mathcal{A} -Hilbert module whose dual can be identified with $H_{-s}(\mathbb{R}^d, \mathcal{W})$. Note also that $H_s^{loc}(\mathbb{R}, \mathcal{W})$ is an \mathcal{A} -module. Extending the classical case $\mathcal{A} = \mathbb{R}$, symbols are defined as follows:

Definition 2.2. (1) *A function $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W}))$ is a symbol of order $m \in \mathbb{R}$, denoted by $a \in S_{\mathcal{W}}^m = S_{\mathcal{W}}^m(\mathbb{R}^d \times \mathbb{R}^d)$, if the following conditions hold:*

(Sy1) *$a(x, \xi)$ has compact support in x ;*

(Sy2) *for any multiindices, α and β , there exists a constant, $C_{\alpha\beta}$, such that*

$$(2.1) \quad \|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\| \leq C_{\alpha\beta} (1 + |\xi|)^{m - |\beta|}.$$

(2) (cf [Sh1]) *A symbol $a \in S_{\mathcal{W}}^m$ is classical if it admits an expansion of the form $\sum_{j \geq 0} \psi(\xi) a_{m-j}(x, \xi)$ where $\psi \in C^\infty(\mathbb{R}^d)$ satisfies*

$$\psi(\xi) = \begin{cases} 0 & \text{for } |\xi| \leq \frac{1}{2} \\ 1 & \text{for } |\xi| \geq 1 \end{cases}$$

(Sy3) $a_{m-j} \in C^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}), \mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W}))$ has compact x -support and is positively homogeneous of degree $m-j$;

(Sy4) $a(x, \xi) - \sum_{j=0}^{N-1} \psi(\xi) a_{m-j}(x, \xi) \in S_{\mathcal{W}}^{m-N}$ for all $N \geq 0$.

Subsequently, we always assume that all symbols are classical.

Given $a \in S_{\mathcal{W}}^m$, define a linear operator $A : C_0^\infty(\mathbb{R}^d, \mathcal{W}) \longrightarrow C_0^\infty(\mathbb{R}^d, \mathcal{W})$ by

$$(2.2) \quad Au(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \int_{\mathbb{R}^d} dx e^{i(x-y, \xi)} a(x, \xi) u(y).$$

The principal symbol of A , $\sigma_A(x, \xi) = a_m(x, \xi)$, is invariantly defined as a smooth function on $T^*\mathbb{R}^d \setminus \{0\}$ with values in $\mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W})$.

The operator A is said to be a pseudodifferential \mathcal{A} -operator of order m , denoted $A \in \Psi DO_{\mathcal{B}}^m(M)$, and can be extended to a bounded, linear operator (any $s \in \mathbb{R}$)

$$A : H_s(\mathbb{R}^d, \mathcal{W}) \longrightarrow H_{s-m}(\mathbb{R}^d, \mathcal{W}).$$

The Schwartz kernel of A , $K_A(x, y)$, is given formally as an oscillatory integral

$$(2.3) \quad K_A(x, y) = \int_{\mathbb{R}^d} e^{i(x-y, \xi)} a(x, \xi) d\xi.$$

To characterize the properties of this distributional kernel we introduce, following [Hö, p.100],

Definition 2.3. A distribution $K \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W}))$ with values in $\mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W})$, is conormal of order m along the diagonal $\text{Diag}(\mathbb{R}^d) = \{(x, x); x \in \mathbb{R}^d\}$, denoted $K \in I^m(\mathbb{R}^d \times \mathbb{R}^d, \text{Diag}(\mathbb{R}^d), \mathcal{L}_{\mathcal{A}}(\mathcal{W}))$, if for all C^∞ -vectorfields V_1, \dots, V_N ($N \geq 1$ arbitrary) which are tangential to $\text{Diag}(\mathbb{R}^d)$ and $\phi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$

$$V_1 \cdots V_N \phi K \in H_{-m-\frac{2d}{4}}(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W})).$$

Following [Hö], one verifies the following

Lemma 2.4. Let $a \in S_{\mathcal{W}}^m$ and let K_A be the corresponding distributional kernel as defined above. Then $K_A \in I^m(\mathbb{R}^d \times \mathbb{R}^d, \text{Diag}(\mathbb{R}^d), \mathcal{L}_{\mathcal{A}}(\mathcal{W}))$.

We note that if $m < -d$, the kernel $K_A(x, y)$ is continuous.

Definition 2.5. A pseudodifferential operator A in $\Psi DO_{\mathcal{W}}^m(\mathbb{R}^d)$ is said to be elliptic on $U_1 \subset \mathbb{R}^d$ if the principal symbol $a_m(x, \xi)$ is invertible for $\xi \in \mathbb{R}^d \setminus \{0\}$ for all $x \in U_1$ and

$$(2.4) \quad \|a_m(x, \xi)^{-1}\| \leq C_1(1 + |\xi|)^{-m} \quad \text{for } x \in U_1, |\xi| \geq 1.$$

Note that as $a_m(x, \xi) \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W})$ the inverse satisfies $a_m(x, \xi)^{-1} \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W})$.

2.2 Pseudodifferential operators on bundles of Hilbert modules.

Throughout this section let (M, g) denote a compact Riemannian manifold of dimension d , possibly with boundary, \mathcal{A} a finite von Neumann algebra, \mathcal{W} an \mathcal{A} -Hilbert module of finite type and $p : \mathcal{E} \rightarrow M$ a bundle of \mathcal{A} -Hilbert modules with fiber \mathcal{W} .

Introduce the Banach bundles of bounded linear operators $\mathcal{L} \rightarrow M \times M$ and $\mathcal{B} = \mathcal{L}_{\mathcal{A}} \rightarrow M \times M$ whose fibres at $(x, y) \in M \times M$ are given by

$$\mathcal{L}_{xy} = \mathcal{L}(\mathcal{E}_y, \mathcal{E}_x); \quad \mathcal{B}_{xy} = \mathcal{B}(\mathcal{E}_y, \mathcal{E}_x).$$

where $\mathcal{L}(\mathcal{E}_y, \mathcal{E}_x)$ denotes the Banach space of all bounded linear operators from the fiber \mathcal{E}_y to the fiber \mathcal{E}_x and

$$\mathcal{B}_{xy} = \{f \in \mathcal{L}_{xy}; f \text{ is } \mathcal{A}\text{-linear}\}.$$

In a straightforward manner one may verify that the Banach bundle $\omega : \mathcal{B} \rightarrow M \times M$ has the following properties

- (Bu1) \mathcal{B}_{xy} is a weakly closed linear subspace of \mathcal{L}_{xy} ;
- (Bu2) if $b \in \mathcal{B}_{xy}$, then $b^* \in \mathcal{B}_{yx}$;
- (Bu3) if $b \in \mathcal{B}_{xy}$, $b' \in \mathcal{B}_{yz}$, then $bb' \in \mathcal{B}_{xz}$;
- (Bu4) $\text{Id} \in \mathcal{B}_{xx}$;
- (Bu5) if $a \in \mathcal{B}_{xx}$ is invertible then $a^{-1} \in \mathcal{B}_{xx}$.

Let U be an open connected subset of M and $X = \mathbb{R}^d$ or, in case U is a neighborhood of a boundary point of M , $X = \mathbb{R}_+^d := \{(x_1, \dots, x_d); x_d \geq 0\}$.

Definition 2.6. A pair (ϕ, Φ) of smooth maps $\phi : U \rightarrow X$ and $\Phi : \mathcal{E}|_U \rightarrow X \times \mathcal{W}$ is said to be a coordinate chart of $(M, \mathcal{E} \rightarrow M)$ if ϕ is a chart of M and Φ is an \mathcal{A} -trivialization of $\mathcal{E} \rightarrow M$ over U .

In particular, $\Phi_x := \Phi|_{p^{-1}(x)} : p^{-1}(x) \rightarrow \mathcal{W}$ is an isometry.

By a standard localizing procedure, one defines the Sobolev spaces $H_s(\mathcal{E}) = H_s(M, \mathcal{E})$ using the definition of Section 2.1 and a smooth partition of unity subordinate to an open cover of M which comes from an atlas of $\mathcal{E} \rightarrow M$. (Equivalently, the Sobolev norms can be defined using a Riemannian metric on M and a connection on \mathcal{E} .) The inner product in $H_s(\mathcal{E})$ will depend on the particular choice of the partition; however a different choice of partition of unity will lead to an equivalent inner product.

The Sobolev s -norm of an element $u \in H_s(\mathcal{E})$ will be denoted by $\|u\|_s$. We point out that for $s \geq t$, $H_s(\mathcal{E})$ imbeds into $H_t(\mathcal{E})$. This embedding, however, is not compact if \mathcal{W} is of infinite dimension (i.e. Rellich's lemma does not hold) even when M is closed.

To simplify the exposition we again assume that M is closed.

Definition 2.7. (1) A linear operator $A : C^\infty(\mathcal{E}) \rightarrow C^\infty(\mathcal{E})$ is an \mathcal{A} -smoothing operator, if A is of the form

$$(Au)(x) = \int_M K_A(x, y)u(y)dy$$

where the Schwartz kernel K_A of A is a smooth section of the bundle $\mathcal{B} \longrightarrow M \times M$.

(2) A linear operator $A : C^\infty(\mathcal{E}) \longrightarrow C^\infty(\mathcal{E})$ is a pseudodifferential \mathcal{A} -operator of order m if for some atlas $(\phi_j, \Phi_j)_{j \in J}$ of $\mathcal{E} \longrightarrow M$, $A = \sum_j A_j + T$ where T is an \mathcal{A} -smoothing operator and the operators A_j are operators with support in the domain of ϕ_j and, when expressed in local coordinates, pseudodifferential \mathcal{A} -operators of order m .

One shows that $A \in \Psi DO_{\mathcal{B}}^m(M)$ is a linear operator, $A : C^\infty(\mathcal{E}) \longrightarrow C^\infty(\mathcal{E})$, which can be extended, for any $s \in \mathbb{R}$, to a bounded linear operator

$$A : H_s(\mathcal{E}) \longrightarrow H_{s-m}(\mathcal{E}).$$

The principal symbol σ_A of A can be defined invariantly as a smooth function $\sigma_A(x, \cdot) : T^*M \setminus \{0\} \longrightarrow \mathcal{L}_{\mathcal{A}}(\mathcal{E}_x)$.

Note that $\Psi DO_{\mathcal{B}}^{-\infty}(M) = \cap_m \Psi DO_{\mathcal{B}}^m(M)$ is the space of \mathcal{A} -smoothing operators. While it is clear that operators in $\Psi DO_{\mathcal{B}}^{-\infty}(M)$ have smooth kernels, it is in general not true that such operators are compact.

As in the classical theory one develops a calculus for these pseudodifferential operators. In particular, one shows that the composition $A \circ B$ of two pseudodifferential operators A and B as well as the adjoint A^* (with respect to the Hermitian structure on $\mathcal{E} \longrightarrow M$) are pseudodifferential operators of the expected order.

2.3 Elliptic pseudodifferential operators.

To simplify the exposition we assume that M is closed.

Definition 2.8. An operator $A \in \Psi DO_{\mathcal{B}}^m(M)$ is said to be elliptic if the principal symbol of A , $\sigma_A(x, \xi)$, is invertible for all $x \in M$ and all $\xi \in T_x^*M \setminus \{0\}$.

As in the classical case one can construct a parametrix, $R(\mu)$, for the operator $(\mu - A)$ when A is elliptic and $\mu \in \mathbb{C} \setminus \bigcup_{(x, \xi) \in T^*M \setminus \{0\}} \text{spec}(\sigma_A(x, \xi))$.

The operator $R(\mu)$ is an element of $\Psi DO_{\mathcal{B}}^{-m}(M)$ and represents an inverse of $(\mu - A)$ up to smoothing operators. Let U be a chart of M which belongs to an atlas of $\mathcal{E} \longrightarrow M$. Denote by ϕ and Φ the diffeomorphisms

$$\begin{aligned} \phi : \mathbb{R}^d &\longrightarrow U \subset M \\ \Phi : \mathbb{R}^d \times \mathcal{W} &\longrightarrow \mathcal{E}|_U \end{aligned}$$

where U is an open subset of M and Φ trivializes the bundle $p : \mathcal{E} \longrightarrow M$ over U such that $p\phi = \Phi p_1$ with $p_1 : \mathbb{R}^d \times \mathcal{W} \longrightarrow \mathbb{R}^d$.

The symbol of $R(\mu)$ in the chart U has an asymptotic expansion determined inductively as follows:

$$r_{-m}(x, \xi, \mu) = (\mu - a_m(x, \xi))^{-1}$$

and, for $j \geq 1$,

(2.5)

$$r_{-m-j}(x, \xi, \mu) = r_{-m}(x, \xi, \mu) \left(\sum_{k=0}^{j-1} \sum_{|\alpha|+l+k=j} \frac{1}{\alpha!} \partial_\xi^\alpha a_{m-l}(x, \xi) D_x^\alpha r_{-m-k}(x, \xi, \mu) \right)$$

where α is a multiindex, $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha! = \alpha_1! \alpha_2! \dots \alpha_d!$, and $D_x^\alpha = (\frac{1}{i} \partial_x)^\alpha$. The term $r_{-m-j}(x, \xi, \mu)$ is an element of $\mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W})$ and is positively homogeneous of degree $-m-j$ in $(\xi, \mu^{\frac{1}{m}})$:

$$r_{-m-j}(x, \lambda \xi, \lambda^m \mu) = \lambda^{-m-j} r_{-m-j}(x, \xi, \mu)$$

for any $\xi \in \mathbb{R}^d \setminus \{0\}$ and $\lambda > 0$.

In the classical case the parametrix of an invertible elliptic pseudodifferential operator is readily used to show that the inverse of an operator is also pseudodifferential. For our more general calculus, additional arguments are necessary.

Proposition 2.9. *Assume that M is closed and that $A \in \Psi DO_{\mathcal{B}}^m(M)$ is elliptic. If A considered as a bounded linear operator, $A : H_m(\mathcal{E}) \longrightarrow L_2(\mathcal{E})$, is one-to-one and onto, then $A^{-1} \in \Psi DO_{\mathcal{B}}^{-m}(M)$.*

Proof: Denote by $B \in \Psi DO_{\mathcal{B}}^{-m}(M)$ a parametrix for A . The operators $T_1 := AB - \text{Id}$ and $T_2 = BA - \text{Id}$ are in $\Psi DO_{\mathcal{B}}^{-\infty}(M)$. From this we conclude that $A^{-1} = B - A^{-1}T_1$. The statement follows once we prove that $A^{-1}T_1 \in \Psi DO_{\mathcal{B}}^{-\infty}(M)$ or, equivalently, that $A^{-1}T_1$ has a smooth Schwartz kernel, $K_{A^{-1}T_1}(x, y) \in \mathcal{B}_{xy} \subset \mathcal{L}(\mathcal{E}_y, \mathcal{E}_x)$. This is proved by using a technique due to Shubin [Sh2]. \diamond

To state Lemma 2.10 we introduce the space $\Psi DO_{\mathcal{L}(\mathcal{E})}^m(M)$ of pseudodifferential operators of order m with coefficients in $\mathcal{L}(\mathcal{E})$. They are defined as were the operators in $\Psi DO_{\mathcal{B}}^m(M)$ by simply replacing the bundle $\mathcal{B} \longrightarrow M \times M$ with the bundle $\mathcal{L}(\mathcal{E}) \longrightarrow M \times M$ whose fibres $\mathcal{L}(\mathcal{E})_{xy}$ are given by $\mathcal{L}(\mathcal{E}_y, \mathcal{E}_x)$.

Lemma 2.10. *Let M be a closed manifold of dimension d . $\Psi DO_{\mathcal{B}}^m(M)$ is a closed subspace in $\Psi DO_{\mathcal{L}(\mathcal{E})}^m(M)$ with respect to the topology provided by the operator norm, $\|A\|$, where $A \in \Psi DO_{\mathcal{L}(\mathcal{E})}^m(M)$ is viewed as a bounded linear operator $A : H_m(\mathcal{E}) \longrightarrow L_2(\mathcal{E})$.*

Proof In view of Lemma 2.9 it suffice to prove the statement for $m < -d$. In that case the result follows by noting that in view of Lemma 2.4 the Schwartz kernel K_A of $A \in \Psi DO_{\mathcal{B}}^m(M)$ is a continuous function of $(x, y) \in M \times M$ and A with $K_A(x, y) \in \mathcal{B}_{xy}$. As \mathcal{B}_{xy} is a closed subspace of $\mathcal{L}(\mathcal{E}_y, \mathcal{E}_x)$, statement (1) follows.

As in the classical theory one proves the following estimates for the resolvent:

Lemma 2.11. (cf [Se1],[Sh1]) Assume that $A \in \Psi DO_{\mathcal{B}}^m(M)$ is an elliptic operator of order $m \geq 0$ such that $A : H_m(\mathcal{E}) \longrightarrow L_2(\mathcal{E})$ is one-to-one and onto. Further assume that π is an Agmon angle for A .

Then for $\lambda < 0$ with $|\lambda|$ sufficiently large and for $0 \leq m' \leq m$,

$$(2.7) \quad |||(\lambda - A)^{-1}|||_{0 \rightarrow m'} \leq C_{m'} |\lambda|^{-1 + \frac{m'}{m}}$$

for some constants $C_{m'} > 0$.

2.4 Zeta-functions and regularized determinants of an invertible elliptic operator.

Let (M, g) be a closed Riemannian manifold. Assume that $A \in \Psi DO_{\mathcal{B}}^m(M)$ is elliptic and of positive order, $m > 0$, with π as an Agmon angle; i.e. there exists $\epsilon > 0$ such that

(1) $V_{\pi, \epsilon} \cap \text{spec} A = \emptyset$;

As a consequence π is also a principal angle

(2) $V_{\pi, \epsilon} \cap \left(\bigcup_{x \in M, (x, \xi) \in S_x^* M} \text{spec}(\sigma_A(x, \xi)) \right) = \emptyset$.

The solid angle $V_{\pi, \epsilon}$ is defined as in the introduction. Note that (1) implies the invertibility of A viewed as a bounded linear operator, $A : H_m(\mathcal{E}) \longrightarrow L_2(\mathcal{E})$. Moreover, for $\Re s < 0$, one can define the complex powers of A by

$$(2.8) \quad A^s := \frac{1}{2\pi i} \int_{\gamma_{\pi, \epsilon}} \mu^{-s} (\mu - A)^{-1} d\mu$$

where $\gamma_{\pi, \epsilon}$ is a path in \mathbb{C} as defined in the introduction. For s satisfying $0 \leq k - 1 \leq \Re s < k \in \mathbb{N}$ one defines

$$A^s = A^k A^{s-k}.$$

It follows from Proposition 2.9 using arguments due to Seeley [Se1], that $A^s \in \Psi DO_{\mathcal{B}}^s(M)$ (after suitably generalizing the concept of order to complex numbers $s \in \mathbb{C}$), depending holomorphically on s . Moreover, for $\Re s < -\frac{d}{m}$, A^s has a von Neumann trace

$$\text{tr}_N(A^s) := \int_M \text{tr}_N K_{A^s}(x, x) dx$$

where $K_{A^s}(x, y) \in \mathcal{L}(\mathcal{E}_y, \mathcal{E}_x)$ denotes the Schwartz kernel of A^s .

For $\alpha \in C^\infty(M, \mathbb{C})$ and $\Re s > \frac{d}{m}$ one defines the generalized zeta-function

$$\zeta_{\alpha, A}(s) = \text{tr}_N(\alpha A^{-s}).$$

As in [Se1] (cf also [Gi] Lemma 1.7.7) one shows

Theorem 2.12. (cf [Se1])

(1) Assume $A \in \Psi DO_{\mathcal{B}}^m(M)$ where $m > 0$ and A is elliptic with π as an Agmon angle. If $\alpha \in C^\infty(M, \mathbb{C})$, then $\zeta_{\alpha, A}(s)$ admits a meromorphic extension to the entire s -plane. The extension has at most simple poles and $s = 0$ is a regular point. The value of $\zeta_{\alpha, A}(s)$ at $s = 0$ is given by

$$(2.9) \quad \zeta_{\alpha, A}(0) = \int_M \alpha(x) I_d(x)$$

where $I_d(x)$ is a density on M . In an appropriate coordinate chart, $I_d(x)$ is given by

$$(2.10) \quad I_d(x) = \frac{1}{m} \frac{1}{(2\pi)^d} \int_{|\xi|=1} d\xi \int_0^\infty \text{tr}_N(r_{-m-d}(x, \xi, -\mu)) d\mu.$$

If A is a differential operator and $d = \dim M$ is odd, then $I_d(x) \equiv 0$.

(2) Assume that $A(t) : H_m(\mathcal{E}) \longrightarrow L_2(\mathcal{E})$ is a family of classical pseudodifferential operators of order m depending in a C^r -fashion on a parameter t varying in an open set of \mathbb{R} . Assume that $A(t)$ is elliptic and that π is an Agmon angle for any t , uniformly in t . Then $\zeta_{A(t)}(s)$ is a family of functions holomorphic in s in a neighborhood of $s = 0$ which depends in a C^r -fashion on t .

Theorem 2.12 above allows us to introduce the ζ -regularized determinant of an elliptic operator $A \in \Psi DO_{\mathcal{B}}^m(M)$ of order $m > 0$ with π as an Agmon angle:

$$(2.11) \quad \det A := \exp \left\{ - \left. \frac{d}{ds} \right|_{s=0} \zeta_A(s) \right\}.$$

To treat the case where A is not invertible, first note the following

Lemma 2.13. Assume $A \in \Psi DO_{\mathcal{B}}^m(M)$. Then the nullspace of A , $\text{Null}(A)$, is an \mathcal{A} -Hilbert module of finite von Neumann dimension, $\dim_N(\text{Null}(A))$.

Assume that A is an elliptic operator, $A \in \Psi DO_{\mathcal{B}}^m(M)$, of order $m > 0$ with π as a weak Agmon angle (i.e $\text{spec}(A) \cup (-\infty, 0) = \emptyset$). Then the operator $A + \lambda$ with $\lambda > 0$ has π as an Agmon angle and $\log \det(A + \lambda)$ is a real analytic function in λ . Define $\log \det_N(A)$ to be the element in \mathbf{D} represented by the analytic function

$$(2.12) \quad \log \det_N(A) := \log \det_N(A + \lambda) - \dim_N(\text{Null}(A)) \log \lambda.$$

Definition 2.14. A is of determinant class if

$$(2.13) \quad \lim_{\lambda \downarrow 0} (\log \det_N(A + \lambda) - \dim_N(\text{Null}(A)) \log \lambda)$$

exists. In this case, $\log \det_N A$ is a real number.

If A is selfadjoint and nonnegative, there is a functional calculus for A . In particular, one can introduce the spectral projections $Q(\lambda)$ corresponding to the intervals $(-\infty, \lambda]$. Using Proposition 2.9 and the assumption that A is nonnegative, one verifies that $Q(\lambda)$ is in $\Psi DO_{\mathcal{B}}^{-\infty}(M)$ for any value of $\lambda \in \mathbb{R}$. Denote the distribution kernel of $Q(\lambda)$ by K_λ and define the spectral function

$$(2.14) \quad N_A(\lambda) := \int_M \operatorname{tr}_N K_\lambda(x, x) dx.$$

Note that $N_A(\lambda)$ is nonnegative, right continuous and monotone increasing as a function of λ . Moreover, $N_A(\lambda) = 0$ for $\lambda < 0$ and for an appropriate constant $C > 0$,

$$N_A(\lambda) \leq C |\lambda|^{\frac{d}{m}}.$$

Proposition 2.15. *Assume that $A \in \Psi DO_{\mathcal{B}}^m(M)$ is an elliptic, selfadjoint, nonnegative operator of order $m > 0$ with π as a principal angle. Then the following statements are equivalent:*

(1) *A is of determinant class.*

(2) $\int_{0+}^1 \log \lambda dN_A(\lambda) > -\infty$.

Here the integral \int_{0+}^1 denotes the Stieltjes integral on the half open interval $(0, 1]$.

The proof of the proposition follows from the heat kernel representation of the determinant which we briefly discuss (cf [Gi]). Let γ be a path in \mathbb{C} defined by the composition $\gamma_- \circ \gamma_+$ of two straight half lines:

$$\gamma_+ := \{x + i(x + 1); -1 \leq x \leq \infty\}$$

$$\gamma_- := \{x - i(x + 1); -1 \leq x \leq \infty\}$$

where γ_+ starts at infinity and γ_- starts at $x = -1$. Using Proposition 2.9 and Lemma 2.10 we may define the following bounded linear operator on $L_2(\mathcal{E})$:

$$e^{-tA} := \frac{1}{2\pi i} \int_\gamma e^{-t\lambda} (\lambda - A)^{-1} d\lambda.$$

One verifies that $e^{-tA} \in \Psi DO_{\mathcal{B}}^{-\infty}(M)$ for $t > 0$. Hence, e^{-tA} has a smooth kernel, denoted $K_A(x, y, t)$, with values in \mathcal{B} and admits a finite von Neumann trace, $\operatorname{tr}_N e^{-tA}$, given by

$$(2.15) \quad \operatorname{tr}_N e^{-tA} = \int_{-\infty}^{\infty} e^{-t\lambda} dN_A(\lambda).$$

As in the classical case one shows that for $t \rightarrow 0$, the kernel $K_A(x, y, t)$ has an expansion on the diagonal $x = y$ of the form

$$(2.16) \quad K(x, x, t) = \sum_{j=0}^{N-1} t^{\frac{j-d}{m}} l_j(x) + O(t^{\frac{N-d}{m}})$$

where $N \geq 1$ is arbitrary and the densities $l_j(x)$, when expressed in local coordinates, are given by expressions of the form (cf [Gi])

$$(2.17) \quad l_j(x) = \frac{1}{2\pi i} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \int_{\gamma} d\mu e^{-\mu} r_{-m-j}(x, \xi, \mu)$$

where $r_{-m-j}(x, \xi, \mu)$, defined inductively by (2.5), are the elements of the symbol expansion of the resolvent $(\mu - A)^{-1}$ of A in local coordinates.

Proof of Proposition 2.15. Using the heat kernel representation of the zeta-function we deduce that

$$\begin{aligned} \log \det(A + \lambda) = & - \left. \frac{d}{ds} \right|_{s=0} \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left(\operatorname{tr}_N e^{-(A+\lambda)t} dt \right) \\ & - \int_1^\infty t^{-1} \left(\operatorname{tr}_N e^{-(A+\lambda)t} dt \right). \end{aligned}$$

The expansion (2.16) is used to show that

$$- \left. \frac{d}{ds} \right|_{s=0} \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left(\operatorname{tr}_N e^{-(A+\lambda)t} - \dim_N(\operatorname{Null}(A)) e^{-\lambda t} \right) dt$$

is a continuous function of λ for $\lambda \geq 0$. To analyze

$$G(\lambda) = \int_1^\infty t^{-1} \left(\operatorname{tr}_N e^{-(A+\lambda)t} - \dim_N(\operatorname{Null}(A)) e^{-\lambda t} \right) dt$$

we write, applying Fubini's theorem together with $\operatorname{tr}_N e^{-At} = \int_{-\infty}^\infty e^{-\mu t} dN_A(\mu)$ and $\dim_N(\operatorname{Null}(A)) = N_A(0)$,

$$\begin{aligned} G(\lambda) &= \int_{0+}^\infty dN_A(\mu) \int_1^\infty t^{-1} e^{-(\mu+\lambda)t} dt \\ &= \int_{\lambda+}^\infty dN_{A+\lambda}(\mu) \int_1^\infty t^{-1} e^{-\mu t} dt. \end{aligned}$$

For $0 < \lambda \leq 1$, write $G(\lambda) = G_1(\lambda) + G_2(\lambda)$ where $G_1(\lambda)$ and $G_2(\lambda)$ are given by

$$\begin{aligned} G_1(\lambda) &= \int_{1+}^\infty dN_{A+\lambda}(\mu) \int_1^\infty t^{-1} e^{-\mu t} dt \\ G_2(\lambda) &= \int_{\lambda+}^1 dN_{A+\lambda}(\mu) \int_1^\infty t^{-1} e^{-\mu t} dt. \end{aligned}$$

The function $G_1(\lambda)$ is estimated in a straightforward way. Concerning $G_2(\lambda)$, note that

$$\int_1^\infty t^{-1} e^{-\mu t} dt = -\log \mu + (1 - e^{-\mu}) \log \mu + \int_\mu^\infty e^{-s} \log s ds$$

and that the function $(1 - e^{-\mu}) \log \mu + \int_\mu^\infty e^{-s} \log s ds$ is bounded for $\mu \in [0, 1]$. \diamond

2.5 Elliptic boundary value problems.

Let (M, g) be a compact Riemannian manifold with boundary $\partial M \neq \emptyset$. For the purpose of this paper we need only consider the Dirichlet problem for an elliptic selfadjoint positive differential operator A of order 2, $A \in DO_{\mathcal{B}}^2(M)$.

Introduce the operator

$$A_D : C_D^\infty(\mathcal{E}) \longrightarrow C^\infty(\mathcal{E})$$

where $C_D^\infty(\mathcal{E}) := \{u \in C^\infty(\mathcal{E}) : u|_{\partial M} = 0\}$. Assume that π is a principal angle for A .

Following [Se2] one constructs a parametrix, $R_D(\mu)$, for $\mu - A_D$ in a similar fashion as in the case $\partial M = \emptyset$, describing inductively the asymptotic expansion of the parametrix symbol. The constructions differ in that, in the case of a manifold with boundary, each term in the expected symbol expansion includes a term arising from the boundary condition. These added terms arising from the boundary conditions only depend on the symbol expansion of A and its derivatives along the boundary ∂M . Having constructed a parametrix one argues as in Proposition 2 of section 2.4 to conclude that $A^{-1} \in \Psi DO_{\mathcal{B}}^{-m}(M)$. This allows one to introduce complex powers of A_D and, for $\Re s > \frac{d}{2}$, the zeta-function $\zeta_{A_D}(s)$ and its generalized version $\zeta_{\alpha, A_D}(s)$ (cf section 2.4). Following Seeley's arguments one obtains the analog of Theorem 2.12

Theorem 2.12'. *Let (M, g) be a compact Riemannian manifold with boundary $\partial M \neq \emptyset$. Assume that A is a selfadjoint, positive, differential operator of order 2 in $DO_{\mathcal{B}}^2(M)$ (note that π is an Agmon angle for A and therefore a principal angle as well). Then the function $\zeta_{\alpha, A_D}(s)$ admits a meromorphic continuation to the entire s -plane. The continuation has at most simple poles and $s = 0$ is a regular point. The value of $\zeta_{\alpha, A_D}(s)$ at $s = 0$ is given by*

$$(2.18) \quad \zeta_{\alpha, A_D}(0) = \int_M \alpha(x) I_d(x) + \int_{\partial M} \alpha(x) B_d(x)$$

where in a coordinate chart of $(M, \mathcal{E} \longrightarrow M)$, $I_d(x)$ is defined as in (2.10). In a coordinate chart of $(\partial M, \mathcal{E}|_{\partial M} \longrightarrow \partial M)$, $B_d(x)$ is given by a formula similiar to that found in [Se2] involving at most the first d terms of the symbol expansion of A and its derivatives up to order d .

Theorem 2.12' allows us to introduce the ζ -regularized determinant of A_D by

$$(2.19) \quad \log \det_N A_D = - \left. \frac{d}{ds} \right|_{s=0} \zeta_{A_D}(s).$$

3. ASYMPTOTIC EXPANSION AND THE MAYER-VIETORIS TYPE FORMULA FOR DETERMINANTS

3.1 1-parameter families with parameter.

Let $\Lambda_{0,\epsilon}$ denote the solid angle in \mathbb{C} given by $\Lambda_{0,\epsilon} = \{re^{i\theta}; r \geq 0, 2\pi|\theta| \leq \epsilon\}$. Consider a family of pseudodifferential operators $A(t)$, $t \in \Lambda_{0,\epsilon}$ with $A(t) \in \Psi DO_{\mathcal{B}}^m(M)$.

Definition 3.1. $A(t)$ is a 1-parameter family of weight χ in $\Psi DO_{\mathcal{B}}^m(M)$ if for any chart $\phi : X \longrightarrow U$ of an atlas of $\mathcal{E} \longrightarrow M$ (where $X = \mathbb{R}^d$, or in case U is a neighborhood of a boundary point, $X = \mathbb{R}_+^d$) and for all $h, h' \in C_0^\infty(U)$, the operator $h'Ah$, when expressed in local coordinates, has an $\mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W})$ -valued symbol $a = a_{h,h';U}$ satisfying the following properties:

(1) for any multiindices α, β there is a constant $C_{\alpha,\beta} > 0$ such that

$$\|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, t)\| \leq C_{\alpha\beta} (1 + |\xi| + |t|^{\frac{1}{\chi}})^{m-|\beta|}$$

where $x \in X$, $\xi \in \mathbb{R}^d$, and $t \in \Lambda_{0,\epsilon}$;

(2) a has an asymptotic expansion

$$(3.1) \quad a \sim \sum_{j \geq 0} \psi(\xi) a_{m-j}(x, \xi, t)$$

with $\psi \in C^\infty(\mathbb{R}^d)$ satisfying

$$\psi(\xi) = \begin{cases} 0 & \text{if } |\xi| \leq \frac{1}{2} \\ 1 & \text{if } |\xi| \geq 1 \end{cases}$$

and $a_{m-j} \in C^\infty(X, \mathbb{R}^d \setminus \{0\}, \Lambda_{0,\epsilon}; \mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W}))$ depending in a C^1 -fashion on the parameter t , has compact x -support and is positive homogeneous of degree $m-j$ in ξ , $t^{\frac{1}{\chi}}$, i.e.

$$a_{m-j}(x, \tau\xi, \tau^{\frac{1}{\chi}}t) = \tau^{m-j} a_{m-j}(x, \xi, t)$$

for all $\tau > 0$.

In the case where M is closed one proves (cf e.g. [Sh1]) that for any $s \in \mathbb{R}$ and $l \geq m$, $A(t)$ is a bounded linear operator, $A(t) : H_s(\mathcal{E}) \longrightarrow H_{s-l}(\mathcal{E})$. Denote by $\|A(t)\|_{s \rightarrow s-l}$ the operator norm of $A(t)$, viewed as an operator $A(t) : H_s(\mathcal{E}) \longrightarrow H_{s-l}(\mathcal{E})$.

Theorem 3.2. *Let M be closed. The following estimates hold:*

- (1) if $l \geq 0$, then $\|A(t)\|_{s \rightarrow s-l} \leq C_{s,l} \left(1 + |t|^{\frac{1}{\chi}}\right)^m$;
- (2) if $m \leq l \leq 0$, then $\|A(t)\|_{s \rightarrow s-l} \leq C_{s,l} \left(1 + |t|^{\frac{1}{\chi}}\right)^{-(l-m)}$.

Definition 3.3. A 1-parameter family $A(t)$ in $\Psi DO_B^m(M)$ is elliptic with parameter, if for any chart $\phi : X \longrightarrow U$ of an atlas of $\mathcal{E} \longrightarrow M$ (where $X = \mathbb{R}^d$, or in case U is a neighborhood of a boundary point, $X = \mathbb{R}_+^d$) and for all $h, h' \in C_0^\infty(U)$, the operator $h'Ah$, when expressed in local coordinates, has principal symbol $a_m(x, \xi, t)$ with values in $\mathcal{L}_A(\mathcal{W}, \mathcal{W})$ such that for all $x \in X$ with $h(\phi(x))h'(\phi(x)) \neq 0$, $a_m(x, \xi, t)$ is invertible for $(\xi, t) \in (\mathbb{R}^d \times \Lambda_{0,\epsilon}) \setminus \{(0, 0)\}$.

Let M be closed. For a 1-parameter family $A(t)$, elliptic with parameter, one constructs a parametrix, $R(\mu, t)$, for $\mu - A(t)$: Given $\mu \notin \bigcup_{t \in \Lambda_{0,\epsilon}} \text{spec}(A(t))$, $R(\mu, t)$ is a 1-parameter family in $\Psi DO_B^{-m}(M)$ satisfying

$$\begin{aligned} R(\mu, t)(\mu - A(t)) - \text{Id} &\in \Psi DO_B^{-\infty}(M) \text{ and} \\ (\mu - A(t))R(\mu, t) - \text{Id} &\in \Psi DO_B^{-\infty}(M). \end{aligned}$$

In local coordinates the symbol of $R(\mu, t)$ is constructed inductively:

$$r_{-m}(x, \xi, t, \mu) = (\mu - a_m(x, \xi, t))^{-1}$$

$$\begin{aligned} (3.2) \quad r_{-m-j}(x, \xi, t, \mu) \\ = r_{-m}(x, \xi, t, \mu) \left(\sum_{k=0}^{j-1} \sum_{|\alpha|+l+k=j} \frac{1}{\alpha!} \partial_\xi^\alpha a_{m-l}(x, \xi, t) D_x^\alpha r_{-m-k}(x, \xi, t, \mu) \right) \end{aligned}$$

where $D_x = \frac{1}{i} \partial_x$. The term $r_{-m-j}(x, \xi, t, \mu)$ is positive homogeneous of degree $-m-j$ in $(\xi, t^{\frac{1}{\chi}}, \mu^{\frac{1}{m}})$.

3.2 Asymptotic expansion for determinants.

As in [BFK2, Appendix], one proves a result concerning the asymptotic expansion of a 1-parameter family $A(t)$ in $\Psi DO_B^m(M)$, $A(t)$ elliptic with parameter.

Theorem 3.4. Let M be a closed manifold. Assume that $A(t)$ is a 1-parameter family in $\Psi DO_B^m(M)$, elliptic with parameter of weight χ and having π as an Agmon angle uniformly in t (cf [BFK1, Theorem 1.1]). Then the function $\log \det_N A(t)$ admits an asymptotic expansion for $t \longrightarrow \infty$ of the form

$$(3.3) \quad \log \det_N A(t) \sim \sum_{-\infty}^d \bar{a}_j |t|^{\frac{j}{\chi}} + \sum_0^d \bar{b}_j |t|^{\frac{j}{\chi}} \log |t|$$

where $\bar{a}_j = \int_M a_j(x, \frac{t}{|t|}) dx$, $\bar{b}_j = \int_M b_j(x, \frac{t}{|t|}) dx$, are defined by smooth densities $a_j(x, \frac{t}{|t|})$ and $b_j(x, \frac{t}{|t|})$, which can be computed in terms of the symbol of $A(t)$.

In particular, with respect to a coordinate chart, $a_0(x, \frac{t}{|t|})$ is given by

$$(3.4) \quad \begin{aligned} a_0 \left(x, \frac{t}{|t|} \right) &= \frac{d}{ds} \Big|_{s=0} \frac{1}{(2\pi)^d} \frac{1}{2\pi i} \int_{\mathbb{R}^d} d\xi \int_{\Gamma} d\mu \mu^{-s} \text{tr}_N \left(r_{-m-d} \left(x, \xi, \frac{t}{|t|}, \mu \right) \right) \\ &= - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \int_0^\infty d\mu \text{tr}_N \left(r_{-m-d} \left(x, \xi, \frac{t}{|t|}, -\mu \right) \right). \end{aligned}$$

A similar result holds in the case where M has a nonempty boundary, $\partial M \neq \emptyset$ (cf [BFK2]). With the notation introduced in section 2.6, one obtains

Theorem 3.5. *Let (M, g) be a compact Riemannian manifold with boundary, $\partial M \neq \emptyset$. Assume that $A(t)$, $t \in \Lambda_{0,\epsilon}$, is a 1-parameter family of selfadjoint, positive differential operators in $\Psi DO_{\mathcal{B}}^m(M)$, of order $m = 2$, elliptic with parameter of weight χ . Assume that there exists $\epsilon' > 0$ such that for all $t \in \Lambda_{0,\epsilon'}$, $\text{spec} A_D(t) \cap V_{\pi,\epsilon'} = \emptyset$. Then the function $\log \det_N A_D(t)$ admits an asymptotic expansion for $t \rightarrow \infty$ of the form*

$$(3.5) \quad \log \det_N A_D(t) \sim \sum_{j=-\infty}^d (\bar{a}_j + \bar{a}_j^b) |t|^{\frac{j}{\chi}} + \sum_{j=0}^d (\bar{b}_j + \bar{b}_j^b) |t|^{\frac{j}{\chi}} \log |t|$$

where \bar{a}_j and \bar{b}_j are given as in Theorem 3.4. The quantities \bar{a}_j^b and \bar{b}_j^b are contributions from the boundary and are of the form

$$(3.6) \quad \bar{a}_j^b = \int_{\partial M} a_j^b \left(x, \frac{t}{|t|} \right); \quad \bar{b}_j^b = \int_{\partial M} b_j^b \left(x, \frac{t}{|t|} \right).$$

In a coordinate chart of $(\partial M, \mathcal{E}|_{\partial M} \rightarrow \partial M)$ the densities $a_j^b(x, \frac{t}{|t|})$ and $b_j^b(x, \frac{t}{|t|})$ are given by a formula each involving only finitely many terms in the symbol expansion of $A(t)$ and finitely many of its derivatives.

3.3 Mayer-Vietoris type formula.

We restrict ourselves to the case needed for this paper. We assume throughout this subsection that (M, g) is a closed Riemannian manifold. Let Γ be a smooth embedded hypersurface in M with trivial normal bundle. Consider an elliptic, selfadjoint, positive, differential operator A of order 2, $A : C^\infty(\mathcal{E}) \rightarrow C^\infty(\mathcal{E})$, of Laplace -Beltrami type (i.e the principal symbol is of the form $\sigma_A(x, \xi) = \|\xi\|^2 Id_{\mathcal{E}_x}$) with $\text{spec}(A) \subset [\epsilon, \infty)$ for some $\epsilon > 0$. Denote by M_Γ the manifold whose interior is $M \setminus \Gamma$, and whose boundary is $\partial M_\Gamma = \Gamma^+ \sqcup \Gamma^-$, where Γ^+ and Γ^- are both copies of Γ . Let g_Γ be the Riemannian metric on M_Γ obtained by pulling back the metric g and let $\mathcal{E}_\Gamma \rightarrow M_\Gamma$ be the pullback of the bundle $\mathcal{E} \rightarrow M$. Consider $A_\Gamma : C^\infty(\mathcal{E}_\Gamma) \rightarrow C^\infty(\mathcal{E}_\Gamma)$ with Dirichlet boundary conditions. Then A_Γ is selfadjoint, positive and elliptic with $\text{spec}(A_\Gamma) \subset [\epsilon, \infty)$ so that π is an Agmon angle for A_Γ . Introduce the Dirichlet to Neumann operator, R_{DN} , associated to the unit vectorfield normal to Γ . This operator is defined as the composition

$$\begin{aligned} C^\infty(\mathcal{E}|_\Gamma) &\xrightarrow{\Delta_{\text{lag}}} C^\infty(\mathcal{E}|_{\Gamma^+}) \oplus C^\infty(\mathcal{E}|_{\Gamma^-}) \xrightarrow{P_D} C^\infty(\mathcal{E}|_\Gamma) \\ &\xrightarrow{N} C^\infty(\mathcal{E}|_{\Gamma^+}) \oplus C^\infty(\mathcal{E}|_{\Gamma^-}) \xrightarrow{\Delta_{\text{diff}}} C^\infty(\mathcal{E}|_\Gamma) \end{aligned}$$

where $\Delta_{\text{diag}}(f) = (f, f)$ is the diagonal operator, P_D is the Poisson operator associated to A_Γ , N is the first order scalar differential operator induced by the normal unit vectorfield along Γ and $\Delta_{\text{diff}}(f^+, f^-) = f^+ - f^-$. As in [BFK2] one proves the following

Theorem 3.6. *Assume that (M, g) is a closed Riemannian manifold and A is an elliptic, selfadjoint, positive differential operator, $A : C^\infty(\mathcal{E}) \longrightarrow C^\infty(\mathcal{E})$ of order 2 of Laplace-Beltrami type with $\text{spec}(A) \subset [\epsilon, \infty)$ for some $\epsilon > 0$. Then R_{DN} is an invertible pseudodifferential operator in $\Psi DO_B^1(\Gamma)$. The inverse R_{DN}^{-1} is given by*

$$(3.7) \quad R_{DN}^{-1} = JA^{-1}(\cdot \otimes \delta_\Gamma)$$

where J is the trace operator $J : H_s(\mathcal{E}) \longrightarrow H_{s-1}(\mathcal{E}|_\Gamma)$ and δ_Γ denotes the Dirac distribution along Γ (cf [BFK2] (4.5)). As a consequence one concludes

- (1) R_{DN} is selfadjoint and positive with $\text{spec}(R_{DN}) \subset [\epsilon', \infty)$ for some $\epsilon' > 0$. In particular, π is an Agmon angle for R_{DN} .
- (2) The principal symbol, $\sigma(R_{DN}^{-1})$, of R_{DN}^{-1} can be computed in terms of the principal symbol $\sigma(A^{-1})$ of A^{-1} (cf [BFK2] (4.6)):

$$(3.8) \quad \sigma(R_{DN}^{-1})(x', \xi') = \frac{1}{2\pi} \int_{\mathbb{R}} \sigma(A^{-1})(x', 0, \xi', \eta) d\eta$$

where $x = (x', w)$ are coordinates in a collar neighborhood of Γ such that x' are coordinates of Γ and the normal vectorfield along Γ is represented by $\frac{\partial}{\partial w}$.

- (3) In a coordinate chart for Γ which arises from a chart belonging to an atlas of $\mathcal{E}|_\Gamma \longrightarrow \Gamma$, the symbol of R_{DN} has an expression whose terms depend only on the terms of the expansion of the symbol of A and its derivatives in an arbitrarily small neighborhood of Γ .
- (4)

$$\det_N(A) = \bar{c} \det_N(A_\Gamma) \det_N(R_{DN})$$

where

$$\bar{c} = \exp \left(\int_\Gamma c(x) \right)$$

and the density $c(x)$, when expressed in a coordinate chart of Γ which is contained in an atlas of $\mathcal{E}|_\Gamma \longrightarrow \Gamma$, depends only on the first d terms of the symbol expansion of A and their derivatives in an arbitrarily small neighborhood of Γ .

- (5) Assume that instead of the single operator A , there is a family $A(t) : C^\infty(\mathcal{E}) \longrightarrow C^\infty(\mathcal{E})$ of differential operators of order 2 of Laplace-Beltrami type with parameter $t \in \Lambda_{0, \epsilon'}$, $\epsilon' > 0$, of weight χ such that $A(t)$ is elliptic, selfadjoint and positive for each t . Introduce as above $A(t)_\Gamma$, $R_{DN}(t)$ and assume that $\text{spec}(A(t)) \cap V_{\pi, \epsilon'} = \emptyset$ for some $\epsilon > 0$ and for all $t \in \Lambda_{0, \epsilon}$. Then $R_{DN}(t)$ is an invertible family of pseudodifferential operators with parameter (cf [BFK2] (3.13)) of order 1 and weight χ .

Remark For the convenience of the reader who is only interested in the results as stated above, Y.Lee has written an easily accessible version of [BFK2] (cf [Lee]).

4.1 Reidemeister and analytic torsion in the von Neumann sense.

Let M be a closed manifold of dimension d and \mathcal{W} an $(\mathcal{A}, \Gamma^{op})$ -Hilbert module of finite type with $\Gamma = \pi_1(M)$ the fundamental group of M . Let $p : \mathcal{E} \rightarrow M$ be the bundle of \mathcal{A} -Hilbert modules over M associated to \mathcal{W} as described in section 1. The fiber of this bundle is isomorphic to the \mathcal{A} -Hilbert module \mathcal{W} . The smooth bundle $p : \mathcal{E} \rightarrow M$ is equipped with a flat *canonical* connection. Both its Hermitian structure μ and fiberwise \mathcal{A} -action ρ are left invariant by the parallel transport induced by the canonical connection.

Let $h : M \rightarrow \mathbb{R}$ be a smooth Morse function. For convenience we assume that h is self-indexing, i.e. $h(x) = \text{index}(x)$ for any critical point x of h . Let g' be a Riemannian metric so that $\tau = (h, g')$ is a generalized triangulation. This means that for any two critical points x and y of h , the unstable manifold W_x^- and the stable manifold W_y^+ , associated to the vector field $\text{grad}_{g'} h$, intersect transversally and, in a neighborhood of any critical point x of h , there exist coordinates y_1, \dots, y_d , with respect to which h is of the form $h(x) = k - (y_1^2 + \dots + y_k^2)/2 + (y_{k+1}^2 + \dots + y_d^2)/2$ with $k = \text{index}(x)$ and the metric g is Euclidean (cf. Introduction). Let $\tilde{M} \rightarrow M$ be the universal covering of M and \tilde{h} and \tilde{g}' be the lifts of h and g' on \tilde{M} . Denote by $\text{Cr}_q(h) \subset M$ resp. $\text{Cr}_q(\tilde{h}) \subset \tilde{M}$ the set of critical points of index q of h resp. \tilde{h} and let $\text{Cr}(\tilde{h}) = \cup_q \text{Cr}_q(\tilde{h})$. Clearly the group Γ acts freely on $\text{Cr}_q(\tilde{h})$, for any q , and the quotient set can be identified with $\text{Cr}_q(h)$.

For each $\tilde{x} \in \text{Cr}(\tilde{h})$ choose an orientation $O_{\tilde{x}}$ for the unstable manifold $W_{\tilde{x}}^-$ and denote $O_h := \{O_{\tilde{x}}; \tilde{x} \in \text{Cr}(\tilde{h})\}$. To the quadruple $(M, \tau, O_h, \mathcal{W})$ we associate a cochain complex of finite type over the von Neumann algebra \mathcal{A} , $\mathcal{C}(M, \tau, O_h) = \{\mathcal{C}^q, \delta_q\}$. The components \mathcal{C}^q are the \mathcal{A} -Hilbert module of finite type, $\mathcal{C}^q := \Gamma(\mathcal{E}|_{\text{Cr}_q(h)}) = \bigoplus_{x \in \text{Cr}_q(h)} \mathcal{E}_x$ which can be identified with the module of Γ -equivariant maps $f : \text{Cr}_q(\tilde{h}) \rightarrow \mathcal{W}$. To define the maps δ_q a few remarks are in order. The orientation on \tilde{M} together with the orientations O_h induce orientations on the stable manifolds $W_{\tilde{x}}^+$ which in turn permit us to define the following functions $m_q : \text{Cr}_q(\tilde{h}) \times \text{Cr}_{q-1}(\tilde{h}) \rightarrow \mathbb{Z}$

$$m_q(\tilde{x}, \tilde{y}) := \text{intersection number } (W_{\tilde{x}}^-, W_{\tilde{y}}^+).$$

Notice that the functions m_q have the following properties:

- (In1) $m_q(\tilde{x}, \tilde{y}) = m_q(g\tilde{x}, g\tilde{y})$, for all $g \in \pi_1(M)$;
- (In2) $\{\tilde{x} \in \text{Cr}_q(\tilde{h}); m_q(\tilde{x}, \tilde{y}) \neq 0\}$ is finite for any $\tilde{y} \in \text{Cr}_{q-1}(\tilde{h})$;
- (In3) $\{\tilde{y} \in \text{Cr}_{q-1}(\tilde{h}); m_q(\tilde{x}, \tilde{y}) \neq 0\}$ is finite for any $\tilde{x} \in \text{Cr}_q(\tilde{h})$;
- (In4) $\sum_{\tilde{y} \in \text{Cr}_{q-1}(\tilde{h})} m_q(\tilde{x}, \tilde{y}) \cdot m_{q-1}(\tilde{y}, \tilde{z}) = 0$ for any $\tilde{x} \in \text{Cr}_q(\tilde{h})$ and any $\tilde{z} \in \text{Cr}_{q-2}(\tilde{h})$.

Properties (In1)-(In3) imply that for any Γ -equivariant map $f : \text{Cr}_{q-1}(\tilde{h}) \rightarrow \mathcal{W}$, we can define the Γ -equivariant map $\delta_{q-1}(f) : \text{Cr}_q(\tilde{h}) \rightarrow \mathcal{W}$ by the formula

$$(4.1) \quad \delta_{q-1}(f)(\tilde{x}) = \sum_{\tilde{y} \in \text{Cr}_{q-1}(\tilde{h})} m_q(\tilde{x}, \tilde{y}) f(\tilde{y}).$$

By property (In4), $\delta_q \cdot \delta_{q-1} = 0$.

One defines $\log T_{\text{comb}}(M, \tau) \in \mathbf{D}$ by

$$(4.2) \quad \log T_{\text{comb}}(M, \tau) := \log T(\mathcal{C}(M, \tau, O_h, \mathcal{W}))$$

(cf section 1).

One can show that $\log T_{\text{comb}}$ is independent of the choice of the orientations O_h .

Let (M, g) be a Riemannian manifold and \mathcal{W} a $(\mathcal{A}, \Gamma^{op})$ -Hilbert module of finite type. Let $\Lambda^q(M; \mathcal{E}) = C^\infty(\mathcal{E} \otimes \Lambda^q(T^*M))$ be the space of smooth q -forms with values in \mathcal{W} where T^*M denotes the cotangent bundle of M and $p : \mathcal{E} \rightarrow M$ is a smooth bundle of \mathcal{A} -Hilbert modules of finite type with fiber \mathcal{W} . The Riemannian metric g induces the Hodge operators $R_q : \Lambda^q(T^*M)_x \rightarrow \Lambda^{d-q}(T^*M)_x$ ($x \in M$) and the Hermitian structure μ on \mathcal{E} together with the Hodge operators induce a Hermitian structure on $\mathcal{E} \otimes \Lambda^q(T^*M)$ given by $(s, s' \in C^\infty(\mathcal{E}); w, w' \in C^\infty(\Lambda^q(T^*M)))$

$$(s \otimes w, s' \otimes w')(x) = \mu_x(s(x), s'(x)) R_q(w(x) \wedge R_q w'(x)).$$

As a consequence $\mathcal{E} \otimes \Lambda^q(T^*M)$ is smooth bundle of \mathcal{A} -Hilbert modules. The canonical connection in $p : \mathcal{E} \rightarrow M$ can be interpreted as a first order differential operator $\mathcal{W}d_q : \Lambda^q(M; \mathcal{E}) \rightarrow \Lambda^{q+1}(M; \mathcal{E})$. As the canonical connection is flat, $\mathcal{W}d_{q+1} \cdot \mathcal{W}d_q = 0$ for any q . Notice that $\mathcal{W}d_q$ is an \mathcal{A} -linear, differential operator and if the action of Γ on \mathcal{W} is trivial, $\mathcal{W}d$ is the usual exterior differential $\text{Id} \otimes d$. In case there is no risk of ambiguity we will write d instead of $\mathcal{W}d$ and continue to call it exterior differential.

The formal adjoint of $\mathcal{W}d_q$ with respect to the above defined Hermitian structure is a first order differential operator $\mathcal{W}d_q^* : \Lambda^{q+1}(M; \mathcal{E}) \rightarrow \Lambda^q(M; \mathcal{E})$ and is again \mathcal{A} -linear. Introduce the Laplacians, acting on q -forms,

$$\Delta_q = d_q^* d_q + d_{q-1} d_{q-1}^*.$$

The operators Δ_q are essentially selfadjoint, nonnegative, elliptic and \mathcal{A} -linear. The space $\Lambda^q(M; \mathcal{E})$ can be equipped with the scalar product

$$(4.3) \quad \langle u_1, u_2 \rangle_r = \langle (\text{Id} + \Delta_q)^{r/2}(u_1), (\text{Id} + \Delta_q)^{r/2}(u_2) \rangle$$

where

$$\begin{aligned} & \langle (\text{Id} + \Delta_q)^{r/2}(u_1), (\text{Id} + \Delta_q)^{r/2}(u_2) \rangle \\ &= \int_M ((\text{Id} + \Delta_q)^{r/2}(u_1), (\text{Id} + \Delta_q)^{r/2}(u_2))(x) d\text{vol}_g. \end{aligned}$$

The completion of $\Lambda^q(M, \mathcal{E})$ with respect to the scalar product $\langle \cdot, \cdot \rangle_r$ is an \mathcal{A} -Hilbert module

$H_r(\Lambda^q(M; \mathcal{E}))$, the space of forms of degree q in Sobolev space of order r . In the case where $r = 0$, we write also $L_2(\Lambda^q(M; \mathcal{E}))$. Obviously, these Hilbert modules are not of

finite type. Note that the operators $(\text{Id} + \Delta_q)^{r/2}$ define isometries between $H_{r'}(\Lambda^q(M; \mathcal{E}))$ and $H_{(r'-r)}(\Lambda^q(M; \mathcal{E}))$. Let \mathcal{H}_q be the \mathcal{A} -Hilbert module of harmonic q -forms

$$\mathcal{H}_q = \{\omega \in L_2(\Lambda^q(M; \mathcal{E})); \Delta_q(\omega) = 0\}.$$

Since Δ_q is elliptic, $\mathcal{H}_q \subset \Lambda^q(M; \mathcal{E})$. The integration $\text{Int}^{(q)}$ on the q -cells of the generalized triangulation τ , which are given by the unstable manifolds of $\text{grad}_g h$, defines an \mathcal{A} -linear map

$$\text{Int}^{(q)} : \Lambda^q(M; \mathcal{E}) \rightarrow \mathcal{C}^q$$

so that $\delta_q \text{Int}^{(q)} = \text{Int}^{(q)} d_q$. Denote by π_q the canonical projection $\pi_q : \mathcal{C}^q \rightarrow \text{Null}(\Delta_q^{\text{comb}})$. By a theorem of Dodziuk [Do] of de-Rham type, the map $\pi_q \text{Int}^{(q)}$, restricted to \mathcal{H}_q , is an isomorphism of Hilbert modules. Denote its inverse by θ_q . Since $\text{Null}(\Delta_q^{\text{comb}})$ is an \mathcal{A} -Hilbert module of finite type so is \mathcal{H}_q . Define T_{met} as the positive real number, viewed as an element in \mathbf{D} (cf. Introduction) by

$$(4.4) \quad \log T_{\text{met}}(M, g, \mathcal{W}, \tau) := \frac{1}{2} \sum_q (-1)^q \log \det_N(\theta_q^* \theta_q).$$

The Reidemeister torsion $T_{\text{Re}}(M, g, \mathcal{W}, \tau) \in \mathbf{D}$ is defined (cf [CM],[LR]) by

$$(4.5) \quad \log T_{\text{Re}}(M, g, \mathcal{W}, \tau) = \log T_{\text{comb}}(M, \mathcal{W}, \tau) + \log T_{\text{met}}(M, g, \mathcal{W}, \tau)$$

and the analytic torsion $T_{\text{an}}(M, g, \mathcal{W}) \in \mathbf{D}$ (cf [Lo],[Ma]) by

$$(4.6) \quad \log T_{\text{an}}(M, g, \mathcal{W}) = \frac{1}{2} \sum_q (-1)^{q+1} q \log \det_N(\Delta_q).$$

Following Gromov-Shubin [GS], for $\lambda \geq 0$, we introduce the functions $F_q(\lambda) := F_{d_q}(\lambda) = \sup\{\dim_N \mathcal{L}; \mathcal{L} \in \mathcal{P}_q(\lambda)\}$ where $\mathcal{P}_q(\lambda)$ consists of all \mathcal{A} -invariant closed subspaces $\mathcal{L} \subset \overline{d_{q-1}(\Lambda^{q-1}(M; \mathcal{E}))} \subset L_2(\Lambda^q(M; \mathcal{E}))$, so that for any $\omega \in \mathcal{L}$, ω is in the domain of definition of d_q and

$$(4.7) \quad \|d_q \omega\| \leq \lambda^{1/2} \|\omega\|.$$

Note that a subspace \mathcal{L} satisfying (4.7) is in fact contained in $\Lambda^{q,+}(M; \mathcal{E})$ where

$$(4.8) \quad \Lambda^{q,+}(M; \mathcal{E}) = \overline{d_{q-1}(\Lambda^{q-1}(M; \mathcal{E}))} \cap \Lambda^q(M; \mathcal{E}).$$

These functions are elements in the space \mathbf{F} (cf section 1). By arguments of Gromov-Shubin which we recalled in section 1.2, the spectral functions $N_k(\lambda) = N_{\Delta_k}(\lambda)$ of the Laplace operator Δ_k are given by $\beta_k + F_{k-1}(\lambda) + F_k(\lambda)$.

Definition 4.1.

(1) The system (M, τ, \mathcal{W}) is said to be of c – determinant class iff for $0 \leq k \leq d$,

$$\int_{0+}^1 \log \lambda dN_{\Delta_k^{\text{comb}}}(\lambda) > -\infty.$$

(2) The system (M, g, \mathcal{W}) is said to be of a – determinant class iff for $0 \leq k \leq d$,

$$\int_{0+}^1 \log \lambda dN_{\Delta_k}(\lambda) > -\infty$$

or, equivalently,

$$\int_0^1 \log \lambda dF_k(\lambda) > -\infty.$$

4.2 Witten's deformation of the analytic torsion.

Let $\omega \in \Lambda^1(M)$ be a smooth closed 1-form on M . Introduce a perturbation $(\Lambda^q(M; \mathcal{E}),_{\mathcal{W}} d_q^\omega)$ of the deRham complex $(\Lambda^q(M; \mathcal{E}),_{\mathcal{W}} d_q)$ with

$$(4.9) \quad d_q^\omega :=_{\mathcal{W}} d_q^\omega :=_{\mathcal{W}} d_q + \omega \wedge (.).$$

The formal adjoint of d_q^ω with respect to the Hermitian structure on $\mathcal{E} \otimes \Lambda^q(T^*M)$, introduced in section 4.1, is a first order \mathcal{A} -linear, differential operator

$$(d_q^\omega)^* : \Lambda^{q+1}(M; \mathcal{E}) \rightarrow \Lambda^q(M; \mathcal{E}).$$

Introduce the perturbed Laplacians, acting on q -forms,

$$(4.10) \quad \Delta_q^\omega = (d_q^\omega)^* d_q^\omega + d_{q-1}^\omega (d_{q-1}^\omega)^*.$$

The operators Δ_q^ω are \mathcal{A} -linear, elliptic operators which are positive and essentially selfadjoint. They are zero'th order perturbations of the Laplacians Δ_q defined above. The case $\omega = tdh$ where $h : M \rightarrow \mathbb{R}$ is a smooth function was considered by Witten *cf* [Wi]. The multiplication by e^{th} defines, for any r , a linear operator on $H_r(\Lambda^q(M; \mathcal{E}))$, which is an isomorphism of \mathcal{A} -Hilbert modules and we have $d_q(t) = e^{-th} d_q e^{th}$. We call the operators $\Delta_q(t) = \Delta_q^{tdh}$ the Witten Laplacians associated to h . More generally, we will refer to the complex $(\Lambda^q(M; \mathcal{E}), d_q^\omega(t))$ with $d_q^\omega(t) = d_q^{t\omega}$ as the Witten complex. Define the perturbed analytic torsion $T_{\text{an}}(M, g, \mathcal{W}, \omega)$ as an element in the vector space \mathbf{D}

$$\log T_{\text{an}}(M, g, \mathcal{W}, \omega) := \frac{1}{2} \sum_q (-1)^q q \log \det_N(\Delta_q^\omega)$$

and the Witten deformation of the analytic torsion $T_{\text{an}}(M, g, \mathcal{W}, \omega)$

$$(4.11) \quad \log T_{\text{an}}(M, g, \mathcal{W}, \omega)(t) := \log T_{\text{an}}(M, g, \mathcal{W}, t\omega).$$

Remark If (M, g, \mathcal{W}) is of a – determinant class and $\omega = dh$ then the Witten deformation satisfies $\log T_{\text{an}}(M, g, \mathcal{W}, t\omega) \in \mathbb{R} \subset \mathbf{D}$, for any t . This can be verified as follows: define functions $F_{\underline{d}_k^{tdh}}(\lambda)$ as above replacing d_k by d_k^{tdh} . As $(L_2(\Lambda^k(M; \mathcal{E})), d_k)$ and $(L_2(\Lambda^k(M; \mathcal{E})), d_k(t))$ are isomorphic, one concludes, according to results of Gromov-Shubin, that $F_{\underline{d}_k^{tdh}}(\lambda) \stackrel{d}{\sim} F_{\underline{d}_k}(\lambda)$ and therefore, as Δ_k is of determinant class, so is $\Delta_k(t)$.

4.3 Product formulas.

For $i = 1, 2$, let \mathcal{A}_i be finite von Neumann algebras, (M_i, g_i, τ_i) closed Riemannian manifolds of dimension d_i (even or odd), equipped with the generalized triangulations $\tau_i = (h_i, g'_i)$. Let \mathcal{W}_i be $(\mathcal{A}_i, \Gamma_i^{op})$ -Hilbert modules of finite type $\Gamma_i = \pi_1(M_i)$, and $\omega_i \in \Lambda^1(M_i)$ closed 1-forms ($i = 1, 2$). Introduce $\mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathcal{W} := \mathcal{W}_1 \otimes \mathcal{W}_2$, $M = M_1 \times M_2$, $g = g_1 \times g_2$, $\tau := (h = h_1 + h_2, g' = g'_1 \times g'_2)$ and $\omega = \omega_1 \otimes 1 + 1 \otimes \omega_2$. Further denote by $\mathcal{E} \rightarrow M$ and $\mathcal{E}_i \rightarrow M_i$ ($i = 1, 2$) the bundles associated to \mathcal{W}_i and \mathcal{W} .

Proposition 4.1(Product formula). *(cf [CM],[Lo],[LR]) With the hypotheses above:*

(1)

$$(4.12) \quad \log T_{\text{an}}(M, g, \mathcal{W}, \omega) = \chi(M_1) \cdot \log T_{\text{an}}(M_2, g_2, \mathcal{W}_2, \omega_2) + \chi(M_2) \cdot \log T_{\text{an}}(M_1, g_1, \mathcal{W}_1, \omega_1)$$

(2)

$$(4.13) \quad \log T_{\text{Re}}(M, g, \mathcal{W}, \tau) = \chi(M_1) \cdot \log T_{\text{Re}}(M_2, g_2, \mathcal{W}_2, \tau_2) + \chi(M_2) \cdot \log T_{\text{Re}}(M_1, g_1, \mathcal{W}_1, \tau_1)$$

Proof: (2) follows from Corolary 1.22 and Proposition 1.9. To prove (1) observe that

$$L_2(\Lambda^r(M, \mathcal{E})) = \oplus_{p+q=r} L_2(\Lambda^p(M_1, \mathcal{E}_1)) \otimes L_2(\Lambda^q(M_2, \mathcal{E}_2))$$

and note that $\Delta_q = \oplus_{p+r=q} \Delta_{(p,r)}$ with

$$\Delta_{(p,r)} = (\Delta'_p \otimes \text{Id}) + (\text{Id} \otimes \Delta''_r)$$

is an \mathcal{A} -linear, elliptic, differential operator where Δ'_p and Δ''_r denote the Laplacians corresponding to $\mathcal{E}_1 \rightarrow M_1$, respectively, $\mathcal{E}_2 \rightarrow M_2$. Notice that $e^{-t\Delta_{p,r}} = e^{-t\Delta'_p} \otimes e^{-t\Delta''_r}$ is of trace class in the von Neumann sense. As in (1.30) introduce

$$(4.14) \quad \zeta_M(\lambda, s) = \frac{1}{2} \sum_{q \geq 1} (-1)^q q \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{tr}_N e^{-t(\Delta_q + \lambda)} dt.$$

As a consequence of a theorem of deRham type due to Dodziuk [Do] one obtains $\chi(M) = \sum (-1)^q \dim_N(\overline{\mathcal{H}}_q(M; \mathcal{E}))$. To prove (1) it suffices to verify that

$$(4.15) \quad \zeta_M(\lambda, s) = \zeta_{M_1}(\lambda, s) \cdot \chi(M_2) + \zeta_{M_2}(\lambda, s) \cdot \chi(M_1).$$

In order to apply the line of arguments of the proof of Proposition 1.21 one needs only to prove that

$$(4.16) \quad \text{tr}_N e^{-t\Delta_{q+1}^+} = \text{tr}_N e^{-t\Delta_q^-}$$

where Δ_q^+ respectively Δ_q^- denote the restriction of Δ_q to $\Lambda^{q,+}(M; \mathcal{E})$ respectively $\Lambda^{q,-}(M; \mathcal{E})$. Equation (4.16) follows from the observation that the spectral projector $P_{q+1}^+(\lambda)$ and $P_q^-(\lambda)$ associated to Δ_{q+1}^+ respectively Δ_q^- are intertwined by d_q and therefore $\text{tr}_N P_{q+1}^+(\lambda) = \text{tr}_N P_q^-(\lambda)$. \diamond

5. Witten's deformation of the deRham complex.

In this section we discuss Witten's deformation of the deRham complex of differential forms with coefficients in a Hilbert bundle $\mathcal{E} \rightarrow M$ of finite type and extend the analysis of Helffer-Sjöstrand [HS1] to this more general setting.

Assume that (M, g) is a closed Riemannian manifold and let $h : M \rightarrow \mathbb{R}$ be a Morse function, so that $\tau = (h, g)$ is a generalized triangulation. Let \mathcal{W} be a $(\mathcal{A}, \Gamma^{op})$ -Hilbert module of finite type with $\Gamma = \pi_1(M)$. To simplify the exposition we assume that \mathcal{W} is a free \mathcal{A} -Hilbert module. Denote by $\mathcal{E} \rightarrow M$ the bundle of \mathcal{A} -Hilbert modules associated to \mathcal{W} . Let $x_{q;j} \in \text{Cr}_q(h)$ be a critical point of index q and U_{qj} an open neighborhood of $x_{q;j}$.

Definition 5.1. U_{qj} is said to be an H -neighborhood of $x_{q;j}$ if there is a ball $B_{2\alpha} := \{x \in \mathbb{R}^d; |x| < 2\alpha\}$ and diffeomorphisms $\phi : B_{2\alpha} \rightarrow U_{qj}$ and $\Phi : B_{2\alpha} \times \mathcal{W} \rightarrow \mathcal{E}|_{U_{qj}}$ with the following properties:

- (i) $\phi(0) = x_{q;j}$;
- (ii) when expressed in the coordinates of ϕ , h is of the form

$$h(x) = q - (x_1^2 + \dots + x_q^2)/2 + (x_{q+1}^2 + \dots + x_d^2)/2;$$

- (iii) the pull back $\phi^*(g)$ of the Riemannian metric g is the Euclidean metric;
- (iv) Φ is a trivialization of $\mathcal{E}|_{U_{qj}}$.

For later use we define $U'_{qj} := \phi(B_\alpha)$.

A collection $(U_x)_{x \in \text{Cr}(h)}$ of H -neighborhoods is called a system of H -neighborhoods if, in addition, they are pairwise disjoint.

As in section 4, denote by $\Lambda^q(M; \mathcal{E}) := C^\infty(\mathcal{E} \otimes \Lambda^q(T^*(M)))$ the \mathcal{A} -module of smooth q -forms with values in \mathcal{E} and by $L_2(\Lambda^q(M; \mathcal{E}))$ its L_2 -completion, which is an \mathcal{A} -Hilbert module. We write $\Lambda^q(M; \mathcal{W})$ for $\Lambda^q(M; \mathcal{E})$ when $\mathcal{E} = M \times \mathcal{W}$ is the trivial bundle and $\Lambda^q(M; \mathbb{R})$ for the space of smooth q -forms on M . Consider the Witten Laplacian $\Delta_q(t) : \Lambda^q(M; \mathcal{E}) \rightarrow \Lambda^q(M; \mathcal{E})$ associated to h and observe that

$$(5.1) \quad \Delta_q(t) = \Delta_q + t^2 \|\nabla h\|^2 + tL_q$$

where L_q is a zero'th order differential \mathcal{A} -operator on $\Lambda^q(M; \mathcal{E})$, hence given by a bundle endomorphism, and where $\|\nabla h\|^2$ is a scalar valued function on M given by $\|\nabla h\|^2(x) = \sum_{1 \leq i, j \leq d} g^{ij}(x) \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j}$ with $(g^{ij}(x))$ denoting the inverse of the metric tensor g when expressed in local coordinates. Δ_q is a nonnegative, selfadjoint, elliptic differential \mathcal{A} -operator. Let $\Lambda^q(M; \mathcal{E})_{\text{sm}}$ be the image (which depends on t) of the spectral projector $Q_q(1, t)$ of $\Delta_q(t)$, corresponding to the interval $(-\infty, 1]$. This space consists of smooth q -forms and is an \mathcal{A} -Hilbert module.

The purpose of this section is to study the complex $(\Lambda^*(M; \mathcal{E})_{\text{sm}}, d_*(t))$ for t sufficiently large and to precisely formulate and prove that this family of complexes converges to the

cochain complex $\mathcal{C}^*(M, \tau, O_h)$, introduced in section 4, when $t \rightarrow \infty$. In the case $\mathcal{A} = \mathbb{R}$ and $\mathcal{W} = \mathbb{R}$ this was done by Helffer and Sjöstrand [HS1]. Their arguments are still valid in the general case. Bismut and Zhang [BZ] verified this in the case $\mathcal{A} = \mathbb{R}$. Here we outline the proof for an arbitrary finite von Neumann algebra \mathcal{A} , referring to [BZ] for those details whose verifications are the same as in the case $\mathcal{A} = \mathbb{R}$.

Consider $h_k : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by $h_k(x) = k + \frac{1}{2}(-\sum_1^k |x_i|^2 + \sum_{k+1}^d |x_i|^2)$ and denote by $\tilde{\Delta}_q : \Lambda^q(\mathbb{R}^d; \mathbb{R}) \rightarrow \Lambda^q(\mathbb{R}^d; \mathbb{R})$ the flat Laplacian on q -forms on \mathbb{R}^d and by $\tilde{\Delta}_{q;k}(t) : \Lambda^q(\mathbb{R}^d; \mathbb{R}) \rightarrow \Lambda^q(\mathbb{R}^d; \mathbb{R})$ the Witten Laplacian associated to h_k . A straightforward calculation shows that

$$(5.2) \quad \tilde{\Delta}_{q;k}(t) = \tilde{\Delta}_q + t^2|x|^2 - t(d - 2k) + 2t(N_{q;k}^+ - N_{q;k}^-),$$

where $N_{q;k}^+$ and $N_{q;k}^-$ are the number operators introduced in [HS1] (cf also [BZ]), defined by

$$N_{q;k}^+(dx_{i_1} \wedge \dots \wedge dx_{i_q}) = \#\{k+1 \leq i_j \leq d\} dx_{i_1} \wedge \dots \wedge dx_{i_q}$$

and $N_{q;k}^- := q\text{Id} - N_{q;k}^+$. Denote by $\tilde{\omega}_q(t) \in \Lambda^q(\mathbb{R}^d; \mathbb{R})$ the q -form defined by

$$(5.3) \quad \tilde{\omega}_q(t) := (t/\pi)^{d/4} e^{-t|x|^2/2} dx_1 \wedge \dots \wedge dx_q.$$

For $\eta > 0$, let $\nu_\eta : \mathbb{R} \rightarrow [0, 1]$ be a smooth map equal to 1 on the interval $(-\infty, \eta/2)$ and equal to 0 on the interval (η, ∞) . For $\epsilon > 0$, which we will choose later at our convenience, define $\tilde{\psi}_q(t) \in \Lambda^q(\mathbb{R}^d; \mathbb{R})$ by

$$(5.3') \quad \tilde{\psi}_q(t) := \beta(t)^{-1} \nu_\epsilon(|x|) \tilde{\omega}_q(t)$$

where $\beta(t) = \|\nu_\epsilon(|x|) \tilde{\omega}_q(t)\|_2$. With respect to the scalar product in $\Lambda^q(\mathbb{R}^d; \mathbb{R})$ induced by the flat metric of \mathbb{R}^d , $\langle \tilde{\omega}_k(t), \tilde{\omega}_k(t) \rangle = 1$ and $\langle \tilde{\psi}_k(t), \tilde{\psi}_k(t) \rangle = 1$. Consider $\Delta_q = \tilde{\Delta}_q \otimes \text{Id}$ and $\Delta_{q;k}(t) = \tilde{\Delta}_{q;k}(t) \otimes \text{Id}$, defined on $\Lambda^q(\mathbb{R}^d; \mathcal{W})$. Both are nonnegative, essentially selfadjoint, elliptic \mathcal{A} -operators and have the following properties:

(HO1) $\text{spec} \Delta_{q;k}(t)$ is discrete and contained in $2t\mathbb{Z}_{\geq 0}$; each eigenvalue has infinite multiplicity if $\dim_{\mathbb{R}} \mathcal{A} = \infty$.

(HO2) $\text{Null}(\Delta_{q;k}(t)) = 0$ if $k \neq q$; $\text{Null}(\Delta_{q;q}(t))$ is an \mathcal{A} -Hilbert module isometric to \mathcal{W} .

(HO3) Assume that $\{v_1, \dots, v_l\}$ is an orthonormal basis of \mathcal{W} , i.e. a collection of orthonormal vectors which generate \mathcal{W} , as an \mathcal{A} -Hilbert module and such that for any $a, b \in \mathcal{A}$,

$$(5.4) \quad \langle av_i, bv_j \rangle = \langle a, b \rangle \delta_{ij}.$$

Then $\omega_{q,i} := \tilde{\omega}_q(t) \otimes v_i$, $1 \leq i \leq l$, is an orthonormal basis for $\text{Null}(\Delta_{q;q}(t))$. Similarly $\psi_{q,i} := \tilde{\psi}_q(t) \otimes v_i$, $1 \leq i \leq l$, provide an orthonormal basis for the \mathcal{A} -Hilbert submodule generated by them. A straightforward calculation, using (5.2), (5.3) and (5.3') and (HO1), shows that there exist constants $C(\epsilon), C_0(\epsilon) > 0$, so that, for $1 \leq i \leq l$, and for t sufficiently large,

$$(5.5) \quad \langle \Delta_{q;q}(t) \psi_{q,i}, \Delta_{q;q}(t) \psi_{q,i} \rangle = O(e^{-C(\epsilon)t}),$$

$$(5.6) \quad \langle \Delta_{q;k}(t) \psi_{q,i}, \psi_{q,i} \rangle \geq C_0(\epsilon)t \quad (k \neq q)$$

and for any $\omega \in \Lambda^q(\mathbb{R}^d; \mathcal{W})$ with $\langle \omega, \psi \rangle = 0$ for ψ in the Hilbert module generated by $\psi_{q,i}, 1 \leq i \leq l$,

$$(5.7) \quad \langle \Delta_{q;q}(t) \omega, \omega \rangle \geq C_0(\epsilon)t \|\omega\|^2.$$

For any two points $y, z \in M$ denote by $d(y, z)$ the distance induced by the metric g and by $d_A(y, z)$ the distance induced by the Agmon metric $g_A = |\nabla h|^2 g$. Let $x \in \text{Cr}_k(h)$ and U_x be an H-neighborhood as defined above. For $y \in U_x$ one has $d_A(x, y) = |y|^2/2$ and $d(x, y) = |y|$. Choose $\epsilon > 0$ so that the balls $B(x; 4\epsilon) = \{y \in M; d(x, y) \leq 4\epsilon\}$, centered at critical points x , are pairwise disjoint, and $B(x; 3\epsilon) \subset U_x$. Choose once and for all a base point $x_0 \in M$, an orthonormal basis e_1, \dots, e_l of \mathcal{E}_{x_0} , and choose for each critical point $x = x_{q;j} \in \text{Cr}_q(h)$ a homotopy class $[\gamma_x]$ of paths, joining x_0 and x (choose $\gamma_{x_0} = \{x_0\}$). Denote by $e_{q;j,1}, \dots, e_{q;j,l}$ the orthonormal basis of \mathcal{E}_x obtained from e_1, \dots, e_l by parallel transport along γ_x , provided by the canonical flat connection on \mathcal{E} . Using the parallel transport, one can identify $\mathcal{E}|_{U_x}$ with $U_x \times \mathcal{W}$ and, using a system of H-neighborhoods U_x , one can identify the forms $\omega \in \Lambda^q(M; \mathcal{E})$ having support in U_x with forms in $\Lambda^q(\mathbb{R}^d; \mathcal{W})$. By extending $\psi_{x,i}(t)$, defined by this identification on U_x , by zero to all of M one obtains a form in $\Lambda^q(M; \mathcal{E})$, which we again denote by $\psi_{x,i}(t)$. The forms $\psi_{x,i}(t) (1 \leq i \leq l, x \in \text{Cr}_q(h))$ satisfy (5.4), and therefore provide an orthonormal basis for the \mathcal{A} -Hilbert module which they generate.

Proposition 5.2. *For any q there exist positive constants C', C'' , and t_0 so that $\text{spec}(\Delta_q(t)) \cap (e^{-tC'}, C''t) = \emptyset$ for $t > t_0$.*

Proof In a first step we prove that for $t > t_0$, with t_0 sufficiently large, there exists a pair of orthogonal closed subspaces $W_1 = W_1(t), W_2 = W_2(t)$ of $L_2(\Lambda^q(M; \mathcal{E}))$ with $W_1 \subset \Lambda^q(M; \mathcal{E})$ so that the following properties hold: (1) $W_1 \cap W_2 = \{0\}$; (2) $W_1 + W_2 = L_2(\Lambda^q(M; \mathcal{E}))$; (3) $\|\Delta_q(t)\omega\| \leq e^{-t2C'} \|\omega\|$ for $\omega \in W_1$; and (4) $\langle \Delta_q(t)\omega, \omega \rangle \geq 2C''t \langle \omega, \omega \rangle$ for $\omega \in W_2 \cap \Lambda^q(M; \mathcal{E})$.

In a second step we show that, using step 1, Proposition 5.2 follows. Let us prove step 2 first. We argue by contradiction. Assume that there exists a sequence $t_j \rightarrow \infty$ and real numbers $\mu_j \in \text{spec} \Delta_q(t_j) \cap (e^{-t_j C'}, C''t_j)$. For each $j \geq 1$, one can find an approximate eigenfunction u_j in the domain of $\Delta_q(t_j)$, $\|u_j\| = 1$, satisfying

$$\|\Delta_q(t_j)u_j - \mu_j u_j\| \leq e^{-4C't_j}.$$

Decomposing $u_j = v_j + w_j \in W_1(t_j) \oplus W_2(t_j)$, one verifies, using the fact that $\Delta_q(t)$ is selfadjoint,

$$|\langle \Delta_q(t_j)u_j, v_j \rangle| \leq \|\Delta_q(t_j)v_j\| \|v_j\| + \|w_j\| \|\Delta_q(t_j)v_j\|$$

as well as

$$\mu_j \|v_j\|^2 = \langle \mu_j u_j, v_j \rangle \leq |\langle \Delta_q(t_j)u_j, v_j \rangle| + |\langle \Delta_q(t_j)u_j - \mu_j u_j, v_j \rangle|.$$

Together with property (3) these two inequalities imply

$$\mu_j \|v_j\| \|w_j\| \leq e^{-2C't_j} (\|v_j\| \|w_j\| + \|w_j\|^2) + e^{-4C't_j} \|w_j\|.$$

It remains to prove step 1. Define $W_1 := W_1(t)$ to be the \mathcal{A} -Hilbert module generated by $\psi_{x,i}(t)$ ($1 \leq i \leq l, x \in \text{Cr}_q(h)$) and $W_2 := W_2(t)$ its orthogonal complement in $L^2(\Lambda^q(M; \mathcal{E}))$. Clearly properties (1) and (2) are satisfied. Further note that an element $\omega \in W_1$ has a representation $\omega = \sum_{1 \leq i \leq l, x \in \text{Cr}_q(h)} a_{x,i} \psi_{x,i}(t)$ with $a_{x,i} \in \mathcal{A}$ and that $\Delta_q(t)$, when restricted to U_x with $x \in \text{Cr}_k(h)$ and expressed in local coordinates introduced above, coincides with $\Delta_{q;k}(t)$. Therefore, in view of (5.4), (5.5) and the support properties of $\psi_{q,i}$, we have, with $C = C(\epsilon)$ as in (5.5),

$$\begin{aligned} \langle \Delta_q(t) \omega, \omega \rangle &= \sum_{1 \leq i \leq l, x \in \text{Cr}_q(h)} \langle a_{x,i} \Delta_q(t) \psi_{q,i}, a_{x,i} \Delta_q(t) \psi_{q,i} \rangle \\ &\leq \sum_{i,x} \|a_{x,i}\|^2 e^{-t2C} \leq \|\omega\|^2 e^{-t2C}. \end{aligned}$$

It remains to check the estimate (4). Denote by $\chi_x : M \rightarrow \mathbb{R}$ the smooth cut-off function with support in U_x defined by $\nu_{2\epsilon}$ and introduce $\chi = \sum_{x \in \text{Cr}(h)} \chi_x$. For $\omega \in W_2 \cap \Lambda^q(M; \mathcal{E})$, define $\omega_1 = \chi \omega$ and $\omega_2 = (1 - \chi) \omega$ and observe that the support of ω_2 is disjoint from the support of any element in W_1 ; therefore $\omega_2 \in W_2 \cap \Lambda^q(M; \mathcal{E})$ and hence $\omega_1 \in W_2 \cap \Lambda^q(M; \mathcal{E})$. Since $\Delta_q(t)$ is essentially selfadjoint one obtains

$$(5.10) \quad \langle \Delta_q(t) \omega, \omega \rangle = \langle \Delta_q(t) \omega_1, \omega_1 \rangle + 2 \langle \Delta_q(t) \omega_1, \omega_2 \rangle + \langle \Delta_q(t) \omega_2, \omega_2 \rangle.$$

We show that there exist positive constants t_0, C_1, C_2, C_3, C_4 depending only on the geometry of $(M, \mathcal{E} \rightarrow M)$ and the chosen ϵ , so that for any $\omega \in W_2 \cap \Lambda^q(M; \mathcal{E})$ and $t > t_0$ the following estimates hold:

$$(5.11) \quad \langle \Delta_q(t) \omega_2, \omega_2 \rangle \geq \langle \Delta_q \omega_2, \omega_2 \rangle + C_1 t^2 \|\omega_2\|^2 - C_2 t \|\omega_2\|^2;$$

$$(5.12) \quad \langle \Delta_q(t) \omega_1, \omega_1 \rangle \geq C_3 t \|\omega_1\|^2;$$

$$(5.13) \quad \langle \Delta_q(t) \omega_1, \omega_1 \rangle \geq \langle \Delta_q \omega_1, \omega_1 \rangle - C_2 t \|\omega_1\|^2.$$

For $\alpha > 0$, $\langle \Delta_q(t) \omega_1, \omega_2 \rangle$ is bounded from below by

$$(5.14) \quad -2C_4(1 + \alpha^{-2})(\|\omega_1\|^2 + \|\omega_2\|^2) - C_4 \alpha^2 \langle \Delta_q \omega_2, \omega_2 \rangle - C_4 \alpha^2 \langle \Delta_q \omega_1, \omega_1 \rangle.$$

Note that (5.12) and (5.13) imply that for any $0 \leq \delta \leq 1$

$$(5.15) \quad \langle \Delta_q(t) \omega_1, \omega_1 \rangle \geq (1 - \delta) \langle \Delta_q \omega_1, \omega_1 \rangle + t(\delta C_3 - (1 - \delta) C_2) \|\omega_1\|^2.$$

To prove (5.11) choose $C_1 := \inf_{z \in M \setminus \cup_{x \in \text{Cr}(h)} U_x} \|\nabla h(z)\|^2$ and $C_2 = \sup_{x \in M} \|L(x)\|$. The estimate (5.11) then follows from (5.1).

To prove (5.12) it suffices to notice that the support of ω_1 is contained in $\cup_{x \in \text{Cr}(h)} U_x$ and ω_1 is orthogonal to $\psi_{x,i}(x \in \text{Cr}_q(h), 1 \leq i \leq l)$. Thus (5.12) follows from (5.7) with $C_3 := C_0(\epsilon)$.

Formula (5.13) is a direct consequence of (5.1).

To find the lower bound (5.14) note that $|\langle L\omega_1, \omega_2 \rangle| \leq C_2 |\langle \omega_1, \omega_2 \rangle|$ and, using that $\text{supp}(\omega_2)$ does not intersect any of the U'_x s, $\langle \|\nabla h\|^2 \omega_1, \omega_2 \rangle \geq C_1 \langle \chi(1 - \chi)\omega, \omega \rangle \geq 0$. Combining with (5.1) one concludes

$$\begin{aligned} \langle \Delta_q(t)\omega_1, \omega_2 \rangle &= \langle \Delta_q \omega_1, \omega_2 \rangle + t^2 \langle \|\nabla h\|^2 \omega_1, \omega_2 \rangle + t \langle L\omega_1, \omega_2 \rangle \\ &\geq \langle \Delta_q \omega_1, \omega_2 \rangle + (C_1 t^2 - C_2 t) \langle \omega_1, \omega_2 \rangle. \end{aligned}$$

For $t > C_2/C_1$ one thus obtains

$$(5.16) \quad \langle \Delta_q(t)\omega_1, \omega_2 \rangle \geq \langle \Delta_q \omega_1, \omega_2 \rangle.$$

Therefore, the lower bound (5.14) follows from Lemma 5.3 below.

To complete the proof of property (4) combine (5.10) with the estimates (5.11), (5.14), and (5.15) to obtain for $0 < \delta < 1$, $\alpha > 0$ and $t > C_2/C_1$,

$$\begin{aligned} \langle \Delta_q(t)\omega, \omega \rangle &\geq (1 - 2C_4\alpha^2) \langle \Delta_q \omega_2, \omega_2 \rangle + (1 - \delta - 2C_4\alpha^2) \langle \Delta_q \omega_1, \omega_1 \rangle \\ &\quad + (C_1 t^2 - C_2 t - 4C_4(1 + \alpha^{-2})) \|\omega_2\|^2 + (t(\delta C_3 - (1 - \delta)C_2) - 4C_4(1 + \alpha^{-2})) \|\omega_1\|^2. \end{aligned}$$

First choose $0 < \delta < 1$ sufficiently close to 1 so that $\delta C_3 - (1 - \delta)C_2 > 0$. Then choose $\alpha > 0$ sufficiently small so that $1 - \delta - 2C_4\alpha^2 > 0$. Together with $2(\|\omega_1\|^2 + \|\omega_2\|^2) \geq \|\omega\|^2$ this establishes property (4) for $t > t_0$ if $t_0 > C_2/C_1$ is chosen sufficiently large. \diamond

Lemma 5.3. *Let the q -forms ω , ω_1 and ω_2 be defined as above. Then there exists a constant $C_4 > 0$ so that, for any $\alpha > 0$,*

$$\langle \Delta_q \omega_1, \omega_2 \rangle \geq -C_4(1 + \alpha^{-2}) \|\omega\|^2 - C_4\alpha^2 \langle \Delta_q \omega_2, \omega_2 \rangle - C_4\alpha^2 \langle \Delta_q \omega_1, \omega_1 \rangle.$$

Proof Write $\Delta_q = d_{q-1}d_{q-1}^* + d_q^*d_q$ where $d_{q-1}^* = -(-1)^{dq}R_{d-q+1}d_{d-q}R_q$, and $*$ = R_q denotes the Hodge $*$ operator. Using that $\omega_1 = \chi\omega$ and $\omega_2 = (1 - \chi)\omega$ one obtains

$$\langle \Delta_q \omega_1, \omega_2 \rangle = \langle d\omega_1, d\omega_2 \rangle + \langle d * \omega_1, d * \omega_2 \rangle \geq A + B,$$

where

$$A := \langle d\chi \wedge \omega, u(1 - \chi)d\omega \rangle + \langle d\chi \wedge * \omega, u(1 - \chi)d * \omega \rangle,$$

$$B := -\langle \chi d\omega, u d\chi \wedge \omega \rangle - \langle \chi d * \omega, u d\chi \wedge * \omega \rangle,$$

where u is the characteristic function of $M \setminus \text{supp} \chi$.

In order to estimate the expressions A and B we introduce the constant $C_5 := \sup_{1 \leq k \leq d} |||K_k|||$, where $K_k : L_2(\Lambda^k(M; \mathcal{E})) \rightarrow L_2(\Lambda^{k+1}(M; \mathcal{E}))$ is the exterior multiplication by $d\chi$. Note that $|||K_k||| = |||K_k^*|||$ where K_k^* denotes the adjoint of K_k . A straightforward calculation yields

$$\begin{aligned} |A| &\leq C_5 ||\omega|| (|(1 - \chi)d\omega| + |(1 - \chi)d * \omega|) \\ &\leq C_5 ||\omega|| (|d\omega_2| + |d\chi \wedge \omega| + |d * \omega_2| + |d\chi \wedge * \omega|) \\ &\leq C_5 ||\omega|| (|d\omega_2| + |d * \omega_2| + 2C_5 ||\omega||) \\ &\leq \sqrt{2}C_5 ||\omega|| \langle \Delta_q \omega_2, \omega_2 \rangle^{1/2} + 2C_5^2 ||\omega||^2. \end{aligned}$$

Thus for any $\alpha > 0$

$$|A| \leq (2C_5^2 + C_5 \alpha^{-2}) ||\omega||^2 + \alpha^2 \langle \Delta_q \omega_2, \omega_2 \rangle.$$

A similar computation leads to

$$|B| \leq (2C_5^2 + C_5 \alpha^{-2}) ||\omega||^2 + \alpha^2 \langle \Delta_q \omega_1, \omega_1 \rangle.$$

Choosing C_4 appropriately leads to the claimed statement. \diamond

Proposition 5.2 yields, for t sufficiently large, a decomposition of $(\Lambda^q(M; \mathcal{E}), d_q(t))$

$$(\Lambda^q(M; \mathcal{E}), d_q(t)) = (\Lambda^q(M; \mathcal{E})_{\text{sm}}, d_q(t)) \oplus (\Lambda^q(M; \mathcal{E})_{\text{la}}, d_q(t))$$

where $\Lambda^q(M; \mathcal{E})_{\text{sm}}$ is the image (depending on t) of $Q(1, t)$, the spectral projection of $\Delta_q(t)$ corresponding to the interval $(-\infty, 1]$, and $\Lambda^q(M; \mathcal{E})_{\text{la}}$ denotes the orthogonal complement of $\Lambda^q(M; \mathcal{E})_{\text{sm}}$. Accordingly, one can decompose $\Delta_q(t) = \Delta_{q, \text{sm}}(t) + \Delta_{q, \text{la}}(t)$ where $\Delta_{q, \text{sm}}(t)$ denotes the restriction of $\Delta_q(t)$ to $\Lambda^q(M; \mathcal{E})_{\text{sm}}$ and, similarly, $\Delta_{q, \text{la}}(t)$ denotes the restriction to $\Lambda^q(M; \mathcal{E})_{\text{la}}$.

Now assume that (M, \mathcal{W}) is of determinant class. Then, for t sufficiently large, $\log \det_N \Delta_q(t)$, $\log \det_N \Delta_q(t)_{\text{sm}}$, and $\log \det_N \Delta_q(t)$ are all real numbers and, accordingly, we write

$$\log T_{\text{an}}(t) = \log T_{\text{sm}}(t) + \log T_{\text{la}}(t).$$

Let $x = x_{q; j} \in \text{Cr}_q(h)$. Choose $\epsilon > 0$ and let $\{e_{q; j, i}\}$ be the orthonormal basis of \mathcal{E}_x as defined above. Define the \mathcal{A} -linear maps $J_x(t) : \mathcal{E}_x \rightarrow L_2(\Lambda^q(M; \mathcal{E}))$ by

$$(5.20) \quad J_x(t) \left(\sum_i a_i e_{q; j, i} \right) := \sum_i a_i \psi_{x, i}.$$

We point out that the $\psi_{x,i}$'s, and thus $J_x(t)$, depend on the choice of ϵ and notice that $J_x(t)$ is an \mathcal{A} -linear isometry. Let $J_q(t) : \sum_{x \in \text{Cr}_q(h)} \mathcal{E}_x \rightarrow L_2(\Lambda^q(M; \mathcal{E}))$ be the sum $J_q(t) := \sum_{x \in \text{Cr}_q(h)} J_x(t)$. As the images of $J_x(t)$ have disjoint support, the map $J_q(t)$ is also an isometry. Recall that we have denoted by $Q_q(1, t)$ the spectral projector of $\Delta_q(t)$ corresponding to the interval $(-\infty, 1]$. Introduce the map

$$(5.21) \quad H_q(t) := (Q_q(1, t)J_q(t))^*(Q_q(1, t)J_q(t)),$$

where $*$ denotes the adjoint of an operator. $H_q(t)$ is a selfadjoint, nonnegative, bounded, \mathcal{A} -linear operator on $\sum_{x \in \text{Cr}_q(h)} \mathcal{E}_x$.

Proposition 5.4. *For $\epsilon > 0$ sufficiently small, there exists a constant $c > 0$ so that*

$$(5.22) \quad (Q_q(1, t)J_q(t)v - J_q(t)v)(x) = O(e^{-ct}\|v\|)$$

uniformly in $x \in M$ and $v \in \sum_{x \in \text{Cr}_q(h)} \mathcal{E}_x$ and

$$(5.23) \quad H_q(t) = \text{Id} + O(e^{-ct}).$$

Observe that the composition $Q_q(1, t)J_q(t)H_q(t)^{-1/2}$ is an \mathcal{A} -linear isometry from the \mathcal{A} -Hilbert module $\sum_{x \in \text{Cr}_q(h)} \mathcal{E}_x$ to $\Lambda^q(M; \mathcal{E})_{\text{sm}}$.

Proof We proceed as in [BZ, p.128]. In view of Proposition 5.2, for $t > t_0$, $Q_q(1, t)$ is given by the Riesz projector

$$(5.24) \quad Q_q(1, t) = \frac{1}{2\pi i} \int_{S^1} (\lambda - \Delta_q(t))^{-1} d\lambda$$

where S^1 is the unit circle in \mathbb{C} , centered at the origin. The operator $Q_q(1, t)J_q(t) - J_q(t)$ can therefore be represented by a Cauchy integral whose integrand is given by

$$(5.25) \quad (\lambda - \Delta_q(t))^{-1}J_q(t) - \lambda^{-1}J_q(t) = \lambda^{-1}(\lambda - \Delta_q(t))^{-1}\Delta_q(t)J_q(t).$$

By Proposition 5.2 there exists, for any Sobolev norm $\|\cdot\|_r$, a constant $c_r > 0$ so that

$$(5.26) \quad \|\Delta_q(t)J_q(t)(v)\|_r = O(e^{-c_r t}\|v\|),$$

uniformly in $v \in \sum_{x \in \text{Cr}_q(h)} \mathcal{E}_x$.

By proceeding as in [BZ, p 128-129] one can show that there exists C_r so that for t sufficiently large and any $\lambda \in S^1$

$$(5.27) \quad \|(\lambda - \Delta_q(t))^{-1}\|_{r \rightarrow r} \leq C_r t^r.$$

Combining (5.26) and (5.27), one obtains, for $c' < c_r$, uniformly for $y \in M$ and $v \in \sum_{x \in \text{Cr}_q(h)} \mathcal{E}_x$,

$$(5.28) \quad \|(\lambda - \Delta_q(t))^{-1}\Delta_q(t)J_q(t)v\|_r = O(e^{-c' t}\|v\|).$$

Choose $r > d/2$ and use the Sobolev embedding theorem to obtain (5.22) from (5.24), (5.25) and (5.28). (5.23) follows immediately from (5.22). \diamond

Let us now consider the cochain complex $\mathcal{C}(M, \tau, O_h, \mathcal{W})$, which has been introduced in section 4. Define $E_{q;j,i} \in \mathcal{C}^q$ for $1 \leq j \leq m_q$ and $1 \leq i \leq l$ by

$$(5.29) \quad E_{q;j,i}(x_{q;j'}) = \begin{cases} e_{q;j,i} & \text{if } j' = j \\ 0 & \text{if } j' \neq j. \end{cases}$$

We see that $E_{q;j,i}$ is bounded as follows: Assume that \mathcal{W}' is a free \mathcal{A} -Hilbert module of finite type with an orthonormal basis v_1, \dots, v_l and $f : \mathcal{W} \rightarrow \mathcal{W}'$ is a bounded, \mathcal{A} -linear map. Then

$$\|f\| \leq l^{1/2} \sup\{\|f(v_i)\|, 1 \leq i \leq l\} \leq l^{1/2}\|f\|.$$

With respect to this basis the differential δ_q can be written as

$$(5.30) \quad \delta_q(E_{q;j,i}) = \sum_{1 \leq j' \leq m_{q+1}, 1 \leq i' \leq l} \gamma_{q;j,i,j'i'} E_{q;j',i'}$$

Denote by \mathcal{G}^q the \mathcal{A} -Hilbert module (which depends on t) generated by $J_q(t)(e_{q;j,i})$ with $1 \leq j \leq m_q, 1 \leq i \leq l$; this is $W_1(t)$ introduced in Proposition 5.2. Recall that, given two closed subspaces W_1 and W_2 , of a Hilbert space, the semi-distance between them is defined by $sdist(W_1, W_2) := |||Prj_{W_1} - Prj_{W_2} \cdot Prj_{W_1}||| = |||Prj_{W_1} - Prj_{W_1} \cdot Prj_{W_2}|||$, where Prj_{W_i} denotes the orthogonal projector on the subspace W_i . Following Helffer and Sjöstrand [HS1] (p 262) we write $A(t) = \tilde{O}(e^{-tc})$ for a quantity $A(t)$ depending on the choice of the ϵ -dependent collection $(U_x)_{x \in \text{Cr}(h)}$ of H -neighborhoods, if for any $\delta > 0$ there exists $\epsilon_\delta > 0$ so that, for any collection $(U_x)_{x \in \text{Cr}(h)}$ of H -neighborhoods with $\epsilon \leq \epsilon_\delta$, $A(t) = O(e^{-t(c-\delta)})$.

Proposition 5.5. $sdist(\mathcal{G}^q, \Lambda^q(M; \mathcal{E})_{\text{sm}}) = \tilde{O}(e^{-tS_q})$ where $S_q = \inf_{x,y \in \text{Cr}_q(h)} d_A(x, y)$.

Here $d_A(x, y)$ denotes the Agmon distance associated to (M, g, h) as defined in [HS1]. Proposition 5.5 is a generalization of Proposition (1.7) of [HS1] or Theorem (8.15) of [BZ] and can be proved in the same way as in [HS1] once one generalizes Proposition 2.5 in [HS2] as follows

Proposition 5.6. *Let \mathcal{K} be an \mathcal{A} -Hilbert module and \mathcal{K}' an \mathcal{A} -Hilbert submodule of \mathcal{K} with orthonormal basis ψ_1, \dots, ψ_N . Suppose that for any $a, b \in \mathcal{A}$, $\langle a\psi_i, b\psi_j \rangle = \langle a, b \rangle \delta_{ij}$. Let $f : \mathcal{K} \rightarrow \mathcal{K}$ be a selfadjoint, nonnegative, \mathcal{A} -linear operator with $(\alpha, \beta) \cap \text{spec}(f) = \emptyset$ for some real numbers α and β , where $0 < \alpha < \beta$. Suppose that $f(\psi_i) = \mu_i \psi_i + r_i$ where $r_i \in \mathcal{K}$ with $\|r_i\| < \epsilon$ and μ_i are real numbers satisfying $0 \leq \mu_i < \alpha$. Let \mathcal{K}_{sm} denote the range of the spectral projection of f associated to the interval $[0, \alpha]$. Then \mathcal{K}_{sm} is an \mathcal{A} -Hilbert module and*

$$sdist(\mathcal{K}', \mathcal{K}_{\text{sm}}) \leq \frac{N^{1/2}\epsilon}{\beta - \alpha}.$$

Proof Write $(f - \lambda)\psi_i = (\mu_i - \lambda)\psi_i + r_i$ and note that for $\lambda \in \mathbb{C} \setminus (\text{spec } f \cup \{\mu_1, \dots, \mu_N\})$

$$(f - \lambda)^{-1}\psi_i = (\mu_i - \lambda)^{-1}\psi_i - (\mu_i - \lambda)^{-1}(f - \lambda)^{-1}r_i.$$

Denote by γ_R the oriented boundary of $[-(\beta - \alpha)/2, (\beta + \alpha)/2] \times i[-R, R]$ and by $P_{\mathcal{K}'}$ the orthogonal projection on \mathcal{K}' . Applying Cauchy's formula one obtains

$$(P_{\mathcal{K}_{\text{sm}}} \psi_i - \psi_i) = -\frac{1}{2\pi i} \int_{\gamma_R} (\mu_i - \lambda)^{-1}(f - \lambda)^{-1}r_i d\lambda.$$

Letting $R \rightarrow \infty$, the above integral becomes

$$\frac{1}{2\pi i} \int_{-(\beta - \alpha)/2 - i\infty}^{-(\beta - \alpha)/2 + i\infty} (\mu_i - \lambda)^{-1}(f - \lambda)^{-1}r_i d\lambda - \frac{1}{2\pi i} \int_{(\alpha + \beta)/2 - i\infty}^{(\alpha + \beta)/2 + i\infty} (\mu_i - \lambda)^{-1}(f - \lambda)^{-1}r_i d\lambda$$

For $\lambda = -(\beta + \alpha)/2 + it$, or $\lambda = (\alpha + \beta)/2 + it$, with $-\infty < t < \infty$, one obtains

$$\|(\mu_i - \lambda)^{-1}(f - \lambda)^{-1}r_i\| \leq \frac{\epsilon}{(\beta - \alpha)^2/4 + t^2}.$$

Hence

$$\|P_{\mathcal{K}_{\text{sm}}} \psi_i - \psi_i\| \leq \frac{\epsilon}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{(\beta - \alpha)^2/4 + t^2} = \frac{\epsilon}{\beta - \alpha}. \diamond$$

Let

$$\begin{aligned} \alpha_q(t) &:= \sup\{0, \text{spec}(Q_q(1, t)\Delta_q(t))\}; \\ \beta_q(t) &:= \inf\{1 + \text{spec}((\text{Id} - Q_q(1, t))\Delta_q(t))\} - 1. \end{aligned}$$

Theorem 5.7. ([HS1],[BZ])

- (1) For $t \rightarrow \infty$, $\alpha_q(t) \rightarrow 0$ and $\beta_q(t) \rightarrow \infty$.
- (2) There exists a constant t_1 so that for $t > t_1$ the elements

$$(5.31) \quad \varphi_{q;j,i}(t) = Q_q(1, t)J_q(t)H_q(t)^{-1/2}(e_{q;j,i})$$

form an orthonormal basis for $\Lambda^q(M; \mathcal{E})_{\text{sm}}$. Hence $\Lambda^q(M; \mathcal{E})_{\text{sm}}$ is a free \mathcal{A} -Hilbert module of rank $l \times \#\text{Cr}_q(h)$.

- (3) There exist $\eta > 0$ and $C > 0$ such that for t sufficiently large, $1 \leq r \leq l$,

$$\sup_{x \in M \setminus U_{qj}} \|\varphi_{q;j,r}\| \leq Ce^{-\eta t}.$$

- (4) Let $W_{q;j}^-$ denote the unstable manifold with respect to the flow corresponding to $\text{grad}_g h$ at the critical point $x_{q;j}$ of h . Choose a system of H -neighborhoods $(U_{x_{q;j}})$ so that $U_{x_{q;j}} \cap W_{q;j'}^- = \emptyset$ for $j' \neq j$. When expressed in local coordinates on $U_{x_{q;j}} \cap W_{q;j}^-$ the q -forms $\varphi_{q;j,i}(t)$ satisfy the following estimate:

$$\varphi_{q;j,i}(t) = (t/4)^{(d/4)} e^{-t|x|^2/2} (dx_1 \wedge \dots \wedge dx_q \otimes e_i + O(t^{-1})).$$

(5) Representing $d_q(t)$ with respect to this basis

$$d_q(t)\varphi_{q;j,i}(t) = \sum_{1 \leq j' \leq m_{q+1}, 1 \leq i' \leq l} \eta_{q;j,i,j'i'}(t)\varphi_{q;j',i'}(t)$$

the coefficients $\eta_{q;j,i,j'i'}$ satisfy

$$\eta_{q;j,i,j'i'}(t) = e^{-t}(t/\pi)^{1/2}(\gamma_{q;j,i,j'i'} + O(t^{-1/2})).$$

Proof Statement (1) follows from Proposition 5.2. Concerning statement (2), note that the \mathcal{A} -Hilbert module W_1 of finite type as defined in the proof of Proposition 5.2 is free, of rank $l \times \#\text{Cr}_q(h)$ and contained in $\Lambda^q(M; \mathcal{E})_{\text{sm}}$. Therefore, should W_1 be not equal to $\Lambda^q(M; \mathcal{E})_{\text{sm}}$ one could conclude that $\Lambda^q(M; \mathcal{E})_{\text{sm}} \cap W_2 \neq \{0\}$, where W_2 is the orthogonal complement of W_1 as defined in the proof of Proposition 5.2. In view of Proposition 5.2 this is, however, not possible. To verify the estimates (3), (4) and (5) one follows the arguments in [HS1] (Proposition 1.7, Theorem 2.5 and Proposition 3.3) or [BZ] (Theorem 8.15, Theorem 8.27, Theorem 8.30). \diamond

We need an application of the above results (cf [BZ]):

Corollary 5.8.

$$\text{Int}^{(q)}(e^{ht}\phi_{q;j,r}(t)) = \left(\frac{t}{\pi}\right)^{(d-2q)/4} e^{qt}(E_{q;j,r} + O(t^{-1})).$$

Proof We must show that for any cell $W_{q;j'}^-$

$$\int_{W_{q;j'}^-} \phi_{q;j,r}(t)e^{ht} = \left(\frac{t}{\pi}\right)^{(d-2q)/4} e^{qt}(\delta_{jj'}e_{q;j,r} + O(t^{-1})).$$

First, note that, due to Theorem 5.7 and to the choice of $U_{qj'}$, it suffices to consider the case where $j = j'$. Moreover, it suffices to estimate

$$\int_{W_{q;j}^- \cap U_{qj}} \phi_{q;j,r}(t)e^{ht}.$$

Note that on $W_{q;j}^- \cap U_{qj}$, the function e^{ht} is of the form

$$e^{ht} = e^{qt}e^{-t(\sum_1^q x_k^2)/2}.$$

By Theorem 5.7, we conclude that

$$\begin{aligned} \int_{W_{q;j}^- \cap U_{qj}} \phi_{q;j,r}(t)e^{ht} &= \left(\frac{t}{\pi}\right)^{d/4} e^{qt} \int_{W_{q;j}^- \cap U_{qj}} e^{-t\sum_1^q x_k^2} (dx_1 \wedge \dots \wedge x_q e_{q;j,r} + O(t^{-1})) \\ &= e^{qt} \left(\frac{t}{\pi}\right)^{d/4} \left(\frac{t}{\pi}\right)^{-q/2} (e_{q;j,r} + O(t^{-1})). \quad \diamond \end{aligned}$$

Finally we state and prove Proposition 2, which is a generalized version of Proposition 1, mentioned in the introduction, and which is due to Gromov-Shubin (cf also [Ef]).

Proposition 2. ([Ef],[GS]) *Let \mathcal{W} be an \mathcal{A} - Hilbert module of finite type, not necessarily free. Then the following statements are true:*

- (1) *Suppose g is a Riemannian metric and $\tau = (h, g')$ is a generalized triangulation of M . Then the system (M, g, \mathcal{W}) is of a – determinant class iff (M, τ, \mathcal{W}) is of c – determinant class.*
- (2) *If M_1 and M_2 are two homotopy equivalent connected manifolds and τ_1 and τ_2 are generalized triangulations of M_1 , respectively M_2 , then $(M_1, \tau_1, \mathcal{W})$ is of c – determinant class iff $(M_2, \tau_2, \mathcal{W})$ is of c – determinant class.*

Proof (1) Results of Novikov-Shubin ([NS1,2]) imply that (M, g, \mathcal{W}) is of a –determinant class iff (M, g', \mathcal{W}) is. Since, for fixed t , multiplication by e^{th} provides a bounded isomorphism of $L_2(\Lambda^*(M; \mathcal{E})) \rightarrow L_2(\Lambda^*(M; \mathcal{E}))$ which intertwines $d_k(t)$ with d_k it follows from results of Gromov-Shubin ([GS]) that (M, g', \mathcal{W}) is of a – *determinant* class

$$\int_{0+}^1 \log \lambda dN_{\Delta_k(t)}(\lambda) > -\infty$$

for all values of t . By Theorem 5.7 and Corollary 5.8 this is equivalent to saying that $(\Lambda^k(M; \mathcal{E})_{\text{sm}}, d_k(t))$ and $(\Lambda^k(M; \mathcal{E})_{\text{sm}}, \tilde{d}_k(t))$ are of determinant class where $\tilde{d}_k(t) := e^{t(\frac{t}{\pi})^{\frac{-1}{2}}} d_k(t)$. Notice that

$$f_k(t) : \Lambda^k(M; \mathcal{E})_{\text{sm}} \rightarrow \mathcal{C}^k,$$

defined by

$$f_k(t) = \left(\left(\frac{\pi}{t} \right)^{\frac{d-2k}{4}} e^{-tk} \right) \text{Int}^{(k)} e^{th}$$

establishes an isomorphism between $(\Lambda^k(M; \mathcal{E})_{\text{sm}}, \tilde{d}_k(t))$ and \mathcal{C}^k and therefore, by Proposition 1.18, one concludes that (M, g', \mathcal{W}) is of a – *determinant* class iff (M, τ, \mathcal{W}) is of c – *determinant* class. Statement (2) is a direct consequence of Proposition 1.18.

6.1 Asymptotic expansion of Witten's deformation of the analytic torsion.

Let (M, g) be a Riemannian manifold with fundamental group $\Gamma = \pi_1(M)$ and $h : M \rightarrow \mathbb{R}$ a Morse function so that $\tau = (h, g)$ is a generalised triangulation. Let \mathcal{A} be a finite von Neumann algebra and \mathcal{W} an $(\mathcal{A}, \Gamma^{op})$ -Hilbert module of finite type. The canonical bundle $p : \mathcal{E} \rightarrow M$ associated to \mathcal{W} is equipped with a canonical flat connection which in turn induces (via parallel transport) a Hermitian structure μ on $\mathcal{E} \rightarrow M$. Throughout this subsection we assume that (M, \mathcal{W}) is of determinant class.

Definition. A function $a : \mathbb{R} \rightarrow \mathbb{R}$ is said to have an asymptotic expansion for $t \rightarrow \infty$ if there exists a sequence $i_1 > i_2 > \dots > i_N = 0$ and constants $(a_k)_{1 \leq k \leq N}$, $(b_k)_{1 \leq k \leq N}$ such that

$$(6.1) \quad a(t) = \sum_{k=1}^N a_k t^{i_k} + \sum_{k=1}^N b_k t^{i_k} \log t + o(1).$$

For convenience we denote by $\text{FT}(a(t))$ the coefficient a_N in the asymptotic expansion of $a(t)$ corresponding to t^0 .

Denote by β_q the Betti numbers and by $\chi(M, \tau) = \sum_q (-1)^q \beta_q$ the Euler-Poincare characteristic of the cochain complex $\mathcal{C}(M, \tau, O_h)$.

Recall that in section 5 we introduced $T_{\text{an}}(h, t)$, $T_{\text{sm}}(h, t)$ and $T_{\text{la}}(h, t)$ and in section 4 we introduced $T_{\text{Re}}(\tau)$, $T_{\text{comb}}(\tau)$ and $T_{\text{met}}(\tau)$. In this section we prove the following

Theorem A. Let (M, g) be a closed Riemannian manifold of odd dimension, \mathcal{W} an $(\mathcal{A}, \Gamma^{op})$ -Hilbert module of finite type with $l = \dim_{\mathbb{N}} \mathcal{W}$ and $h : M \rightarrow \mathbb{R}$ a Morse function. Assume that (M, \mathcal{W}) is of determinant class and that $\tau = (h, g)$ is a generalized triangulation. Then the following statements are true:

- (1) The functions $\log T_{\text{an}}(h, t)$, $\log T_{\text{sm}}(h, t)$ and $\log T_{\text{la}}(h, t)$ admit asymptotic expansions for $t \rightarrow \infty$.
- (2) The asymptotic expansion of $\log T_{\text{an}}(h, t)$ is of the form

$$(6.2) \quad \log T_{\text{an}}(h, t) = \log T_{\text{an}}(h, 0) - \log T_{\text{met}}(\tau) + \frac{1}{2} \left(\sum_{q=0}^d (-1)^{q+1} q \beta_q \right) (2t - \log t + \log \pi) + O(t^{-1}).$$

- (3) The asymptotic expansion of $\log T_{\text{sm}}(h, t)$ is of the form

$$(6.3) \quad \log T_{\text{comb}}(\tau) + \frac{1}{2} \left(\sum_{q=0}^d (-1)^{q+1} (q \beta_q - q m_q l) \right) (2t - \log t + \log \pi) + o(1).$$

As argued in Introduction it suffices to prove the statements for \mathcal{W} a free \mathcal{A} -module. We begin by deriving an alternative formula for the analytic torsion (cf [RS] ,[Ch] and [BFK1]). The space of q -forms can be decomposed into orthogonal subspaces:

$$(6.4) \quad \Lambda^q(M; \mathcal{E}) = \Lambda_t^{+,q}(M; \mathcal{E}) \oplus \Lambda_t^{-,q}(M; \mathcal{E}) \oplus \mathcal{H}_t^q$$

where

$$(6.5) \quad \Lambda_t^{+,q}(M; \mathcal{E}) = \text{closure}(d_{q-1}(t)\Lambda^{q-1}(M; \mathcal{E}));$$

$$(6.6) \quad \Lambda_t^{-,q}(M; \mathcal{E}) = \text{closure}(d_q(t)^*\Lambda^{q+1}(M; \mathcal{E}));$$

$$(6.7) \quad \mathcal{H}_t^q = \{\omega \in \Lambda^q(M; \mathcal{E}); \Delta_q(t)\omega = 0\}.$$

Note that the spaces $\Lambda_t^{\pm,q}(M; \mathcal{E})$ are invariant with respect to the Laplacian $\Delta_q(t)$. Denote by $\Delta_q^{\pm}(t)$ the restriction of $\Delta_q(t)$ to $\Lambda_t^{\pm,q}(M; \mathcal{E})$ which are given by $\Delta_q^+(t) = d_{q-1}(t)d_{q-1}(t)^*$ and $\Delta_q^-(t) = d_q(t)^*d_q(t)$. The operator $d_q(t)$ maps the space $\Lambda_t^{-,q}(M; \mathcal{E})$ injectively onto a dense subspace of $\Lambda_t^{+,q+1}(M; \mathcal{E})$ and it intertwines $\Delta_q^-(t)$ and $\Delta_{q+1}^+(t)$. As a consequence, $d_q(t)$ intertwines the spectral projectors $Q_q^-(\lambda, t)$ and $Q_{q+1}^+(\lambda, t)$,

$$(6.8) \quad d_q(t)Q_q^-(\lambda, t) = Q_{q+1}^+(\lambda, t)d_q(t).$$

This implies

$$(6.9) \quad \begin{aligned} N_q^-(\lambda, t) &= \text{tr}_N(Q_q^-(\lambda, t)) \\ &= \text{tr}_N(Q_{q+1}^+(\lambda, t)) = N_{q+1}^+(\lambda, t). \end{aligned}$$

Note that both $\Delta_q^+(t)$ and $\Delta_q^-(t)$ are of determinant class i.e $\int_{0+}^1 \log \lambda dN_A^\epsilon(\lambda) > -\infty$, $\epsilon = +, -$. Using the heat kernel representation of the zeta function we obtain (cf [Lo]):

$$(6.10.A) \quad \int_1^\infty \frac{dx}{x} \text{tr}_N \left(e^{-x\Delta_q^-(t)} \right) = \int_1^\infty \frac{dx}{x} \text{tr}_N \left(e^{-x\Delta_{q+1}^+(t)} \right)$$

and for $\Re s$ sufficiently large,

$$(6.10.B) \quad \frac{1}{\Gamma(s)} \int_0^1 dx x^{s-1} \text{tr}_N \left(e^{-x\Delta_q^-(t)} \right) = \frac{1}{\Gamma(s)} \int_0^1 dx x^{s-1} \text{tr}_N \left(e^{-x\Delta_{q+1}^+(t)} \right).$$

Formulas (6.10) are now used to write

$$(6.11) \quad \begin{aligned} \log T_{\text{an}}(h, t) &= \frac{1}{2} \sum_{q=0}^d (-1)^q \log \det_N \Delta_q^-(t) \\ &= \frac{1}{2} \sum_{q=0}^d (-1)^q \log \det_N \Delta_{q+1}^+(t). \end{aligned}$$

Our first goal is to compute the variation $\frac{d}{dt} \log T_{\text{an}}(h, t)$ of $\log T_{\text{an}}(h, t)$. For this purpose, we again use the heat kernel representation of the zeta function and write (cf [Lo])

$$(6.12) \quad \log \det_N \Delta_q^+(t) = - \left. \frac{\partial}{\partial s} \right|_{s=0} \left(\frac{1}{\Gamma(s)} \int_0^1 x^{s-1} \text{tr}_N \left(e^{-x \Delta_q^+(t)} \right) dx \right) \\ - \int_1^\infty \frac{1}{x} \text{tr}_N \left(e^{-x \Delta_q^+(t)} \right) dx.$$

To analyze the t -dependence of $\log \det_N \Delta_q^+(t)$ we treat the two terms on the right hand side of (6.12) separately. To illustrate the new difficulties which arise (as compared with the classical situation) we point out that the differentiability of $\int_1^\infty x^{-1} \text{tr}_N e^{-x \Delta_q^+(t)} dx$ with respect to t is far from being obvious.

We begin by computing $\frac{d}{dt} (\text{tr}_N e^{-x \Delta_q^+(t)})$ and note that $\Delta_q^+(t) : \Lambda_t^{+,q}(M; \mathcal{E}) \rightarrow \Lambda_t^{+,q}(M; \mathcal{E})$ where the space $\Lambda_t^{+,q}(M; \mathcal{E}) = \overline{d_{q-1}(t) \Lambda^{q-1}(M; \mathcal{E})} = e^{-th} \Lambda^{+,q}(M; \mathcal{E})$ depends on t . It is therefore convenient to introduce $\widetilde{\Delta}_q^+(t) = e^{th} \Delta_q^+(t) e^{-th} : \Lambda^{+,q}(M; \mathcal{E}) \rightarrow \Lambda^{+,q}(M; \mathcal{E})$ which is isospectral with $\Delta_q^+(t)$. Hence, $\text{tr}_N e^{-x \Delta_q^+(t)} = \text{tr}_N e^{-x \widetilde{\Delta}_q^+(t)}$. Now one computes $\frac{d}{dt} \text{tr}_N e^{-x \widetilde{\Delta}_q^+(t)}$ using Duhamel's principle and the identity $\widetilde{\Delta}_q^+(t) = e^{2th} (d_{q-1} d_{q-1}^* + 2t dh \wedge d_{q-1}^*) e^{-2th}$:

$$\begin{aligned} \frac{d}{dt} \left(\text{tr}_N e^{-x \widetilde{\Delta}_q^+(t)} \right) &= -x \text{tr}_N \left(\frac{d}{dt} \left(\widetilde{\Delta}_q^+(t) e^{-x \widetilde{\Delta}_q^+(t)} \right) \right) \\ &= \text{tr}_N \left(2[h, -x \widetilde{\Delta}_q^+(t)] e^{-x \widetilde{\Delta}_q^+(t)} \right) \\ &\quad - 2x \text{tr}_N \left(e^{2th} dh \wedge d_{q-1}^* e^{-2th} e^{-x \widetilde{\Delta}_q^+(t)} \right) \end{aligned}$$

where $[A, B]$ denotes the commutator of the two operators A and B . Therefore

$$\text{tr}_N \left(2[h, -x \widetilde{\Delta}_q^+(t)] e^{-x \widetilde{\Delta}_q^+(t)} \right) = 0.$$

Using that $e^{th} d_{q-1}^* e^{-th} = d_{q-1}(t)^*$ and that $e^{-th} e^{-x \widetilde{\Delta}_q^+(t)} e^{th} = e^{-x \Delta_q^+(t)}$ we obtain

$$\frac{d}{dt} \left(\text{tr}_N \left(e^{-x \Delta_q^+(t)} \right) \right) = -2x \text{tr}_N \left(dh \wedge d_{q-1}(t)^* e^{-x \Delta_q^+(t)} \right).$$

Further observe that, despite the fact that $d_q(t) : \Lambda_t^{+,q}(M; \mathcal{E}) \rightarrow \Lambda_t^{-,q+1}(M; \mathcal{E})$ is not invertible (it might not be onto) we can form

$$d_{q-1}(t)^* = d_{q-1}(t)^{-1} d_{q-1}(t) d_{q-1}(t)^* = d_{q-1}(t)^{-1} \Delta_q^+(t)$$

where the domain of definition of $d_{q-1}(t)^{-1}$ is the range of $d_{q-1}(t)$. We note that

$$dh \wedge d_{q-1}(t)^{-1} \Delta_q^+(t) = (d_{q-1}(t) h d_{q-1}(t)^{-1} - h) \Delta_q^+(t).$$

This leads to the following formula

$$\begin{aligned} \frac{d}{dt} \left(\text{tr}_N \left(e^{-x\Delta_q^+(t)} \right) \right) &= -2x \text{tr}_N \left(d_{q-1}(t) h d_{q-1}(t)^{-1} \Delta_q^+(t) e^{-x\Delta_q^+(t)} \right) \\ &\quad + 2x \text{tr}_N \left(h \Delta_q^+(t) e^{-x\Delta_q^+(t)} \right). \end{aligned}$$

Next we observe that

$$2x \text{tr}_N \left(h \Delta_q^+(t) e^{-x\Delta_q^+(t)} \right) = -2x \frac{d}{dx} \left(\text{tr}_N \left(h e^{-x\Delta_q^+(t)} \right) \right)$$

and that

$$\begin{aligned} \text{tr}_N \left(d_{q-1}(t) h d_{q-1}(t)^{-1} \Delta_q^+(t) e^{-x\Delta_q^+(t)} \right) &= \text{tr}_N \left(h d_{q-1}(t)^{-1} \Delta_q^+(t) e^{-x\Delta_q^+(t)} d_{q-1}(t) \right) \\ &= \text{tr}_N \left(h \Delta_{q-1}^-(t) e^{-x\Delta_{q-1}^-(t)} \right) \\ &= -\frac{d}{dx} \left(\text{tr}_N \left(h e^{-x\Delta_{q-1}^-(t)} \right) \right). \end{aligned}$$

We have therefore proved that

$$\frac{d}{dt} \left(\text{tr}_N \left(e^{-x\Delta_q^+(t)} \right) \right) = 2x \frac{d}{dx} \left(\text{tr}_N \left(h e^{-x\Delta_{q-1}^-(t)} \right) \right) - 2x \frac{d}{dx} \left(\text{tr}_N \left(h e^{-x\Delta_q^+(t)} \right) \right).$$

This leads to

$$\begin{aligned} \frac{1}{2} \sum_{q=0}^d (-1)^{q+1} \frac{d}{dt} \left(\text{tr}_N \left(e^{-x\Delta_q^+(t)} \right) \right) &= x \sum_{q=0}^d (-1)^{q+1} \frac{d}{dx} \left(\text{tr}_N \left(h e^{-x\Delta_{q-1}^-(t)} \right) \right) \\ &\quad - x \sum_{q=0}^d (-1)^{q+1} \frac{d}{dx} \left(\text{tr}_N \left(h e^{-x\Delta_q^+(t)} \right) \right) \\ (6.13) \quad &= x \frac{d}{dx} \left(\sum_{q=0}^d (-1)^q \text{tr}_N \left(h e^{-x\Delta_q(t)} (\text{Id} - Q_q(0, t)) \right) \right). \end{aligned}$$

The above formula is used to prove that $-\int_1^\infty \frac{1}{x} \frac{1}{2} \sum_q (-1)^{q+1} \text{tr}_N \left(e^{-x\Delta_q^+(t)} \right) dx$ has a continuous derivative with respect to t .

By the Leibniz rule for improper integrals, it suffices to verify that $f(x, t) = -\frac{1}{x} \frac{1}{2} \sum_q (-1)^{q+1} \text{tr}_N \left(e^{-x\Delta_q^+(t)} \right)$ and $\frac{\partial f}{\partial t}(x, t)$ are both continuous and the integrals $\int_1^\infty f(x, t) dx$ and $\int_1^\infty \frac{\partial f}{\partial t}(x, t) dx$ both converge uniformly with respect to t (t varying in a compact interval). Clearly $f(x, t)$ is continuous and, by the above formula,

$$\frac{\partial f}{\partial t}(x, t) = -\frac{d}{dx} \sum_{q=0}^d (-1)^q \text{tr}_N \left(h e^{-x\Delta_q(t)} (\text{Id} - Q_q(0, t)) \right).$$

The uniform convergence of the integrals $\int_1^\infty f(x, t) dx$ and $\int_1^\infty \frac{\partial f}{\partial t}(x, t) dx$ follows from

Lemma 6.1. *Let I be an arbitrary compact interval contained in $[0, \infty)$. Then*

- (1) $\lim_{x \rightarrow \infty} \int_1^x \frac{1}{x} \text{tr}_N(e^{-x\Delta_q^+(t)}) dx$ converges uniformly for $t \in I$.
- (2) $\lim_{x \rightarrow \infty} \int_1^x \frac{\partial f}{\partial t}(x, t) dx$ converges uniformly for $t \in I$.

Proof (1) Note that the integrand $\frac{1}{x} \text{tr}_N(e^{-x\Delta_q^+(t)})$ is positive. Therefore

$$\begin{aligned}
 (6.14) \quad 0 &\leq \int_u^\infty \frac{1}{x} \text{tr}_N(e^{-x\Delta_q^+(t)}) dx = \int_{0+}^\infty dN_{\Delta_q^+(t)}(\mu) \int_u^\infty \frac{1}{x} e^{-\mu x} dx \\
 &\leq \int_{u^{-\frac{1}{2}}}^\infty dN_{\Delta_q^+(t)}(\mu) \frac{e^{-\mu u}}{\mu u} + \int_{0+}^{u^{-\frac{1}{2}}} dN_{\Delta_q^+(t)}(\mu) \left(\log(\mu u) e^{-\mu u} + \int_{\mu u}^\infty e^{-s} \log s ds \right) \\
 &\leq \frac{e^{-u^{\frac{1}{2}}}}{u^{\frac{1}{2}}} \int_{0+}^\infty dN_{\Delta_q^+(t)}(\mu) + C \int_{0+}^{u^{-\frac{1}{2}}} dN_{\Delta_q^+(t)}(\mu)
 \end{aligned}$$

where $C > 0$ is a bound for the function $\log(\mu u) e^{-\mu u} + \int_{\mu u}^\infty e^{-s} (\log s) ds$ for μ in the interval $[0, 1]$. Statement (1) follows from (6.14) for $u \rightarrow \infty$ and from the fact that $N_{\Delta_q^+(t)}(\mu)$ is right continuous with respect to μ , uniformly in t for t in I .

(2) From (6.13), $\int_1^u \frac{\partial f}{\partial t}(x, t) = -\sum_{q=0}^d (-1)^q \text{tr}_N(h e^{-x\Delta_q^+(t)} (\text{Id} - Q_q(0, t)))|_1^u$. Therefore it suffices to prove that, for $0 \leq q \leq d$ and uniformly for t in I ,

$$(6.15) \quad \lim_{x \rightarrow \infty} \text{tr}_N(h e^{-x\Delta_q(t)} (\text{Id} - Q_q(0, t))) = 0.$$

This can be seen as follows:

$$\begin{aligned}
 |\text{tr}_N(h e^{-x\Delta_q(t)} (\text{Id} - Q_q(0, t)))| &\leq \|h\|_{L^\infty} \int_{0+}^\infty e^{-x\lambda} dN_q(\lambda, t) \\
 &\leq \|h\|_{L^\infty} \left(\int_{0+}^{x^{-\frac{1}{2}}} e^{-x\lambda} dN_q(\lambda, t) + \int_{x^{-\frac{1}{2}}}^\infty e^{-x\lambda} dN_q(\lambda, t) \right) \\
 &\leq \|h\|_{L^\infty} \left((N_q(x^{-\frac{1}{2}}, t) - N_q(0, t)) + e^{-x^{\frac{1}{2}}} \int_0^\infty dN_q(\lambda, t) \right)
 \end{aligned}$$

and (6.15) follows from the fact that $N_q(\lambda, t)$ is right continuous with respect to μ , uniformly for t in I .

We have shown that $-\int_1^\infty \frac{1}{x} \frac{1}{2} \sum_q (-1)^{q+1} \text{tr}_N(e^{-x\Delta_q^+(t)}) dx$ has a continuous derivative with respect to t :

$$\begin{aligned}
 (6.16) \quad \frac{d}{dt} \left(-\int_1^\infty \frac{1}{x} \frac{1}{2} \sum_q (-1)^{q+1} \text{tr}_N(e^{-x\Delta_q^+(t)}) dx \right) \\
 = \sum_{q=0}^d (-1)^q \text{tr}_N(h e^{-x\Delta_q(t)} (\text{Id} - Q_q(0, t))) \cdot \diamond
 \end{aligned}$$

We now analyze the t -derivative of $\frac{\partial}{\partial s}|_{s=0} \frac{1}{\Gamma(s)} \int_0^1 x^{s-1} \frac{1}{2} \sum_q (-1)^{q+1} \text{tr}_N(e^{-x\Delta_q^+(t)}) dx$.

For $\Re s > \frac{d}{2}$, we integrate by parts in (6.12) to obtain

$$\begin{aligned} & -\frac{d}{dt} \left(\frac{1}{\Gamma(s)} \int_0^1 x^{s-1} \frac{1}{2} \sum_q (-1)^{q+1} \text{tr}_N \left(e^{-x\Delta_q^+(t)} \right) dx \right) \\ &= -\frac{1}{\Gamma(s)} \int_0^1 x^s \frac{d}{dx} \left(\sum_{q=0}^d (-1)^q \text{tr}_N \left(h e^{-x\Delta_q(t)} (\text{Id} - Q_q(0, t)) \right) dx \right) \\ &= -\frac{1}{\Gamma(s)} \sum_{q=0}^d (-1)^q \text{tr}_N \left(h e^{-\Delta_q(t)} (\text{Id} - Q_q(0, t)) \right) \\ &\quad + \frac{s}{\Gamma(s)} \int_0^1 x^{s-1} \frac{d}{dx} \sum_{q=0}^d (-1)^q \text{tr}_N (h e^{-x\Delta_q(t)} (\text{Id} - Q_q(0, t))) dx. \end{aligned}$$

The last two functions both have a meromorphic extension to the s -plane which is regular at $s = 0$. With $\frac{1}{\Gamma(s)} = \frac{s}{\Gamma(s+1)}$ and $\Gamma(1) = 1$, we obtain

$$\begin{aligned} (6.17) \quad & \frac{d}{dt} \left(-\frac{\partial}{\partial s}|_{s=0} \frac{1}{\Gamma(s)} \int_0^1 x^{s-1} \frac{1}{2} \sum_q (-1)^{q+1} \text{tr}_N \left(e^{-x\Delta_q^+(t)} \right) dx \right) \\ &= -\sum_{q=0}^d (-1)^q \text{tr}_N \left(h e^{-\Delta_q(t)} (\text{Id} - Q_q(0, t)) \right) \\ &\quad + \text{F.p.}_{s=0} \frac{1}{\Gamma(s)} \int_0^1 x^{s-1} \sum_{q=0}^d (-1)^q \text{tr}_N \left(h e^{-x\Delta_q(t)} (\text{Id} - Q_q(0, t)) \right) dx. \end{aligned}$$

Combing (6.16) and (6.17) we conclude that $\log T_{\text{an}}(h, t)$ is continuous and has a continuous derivative with respect to t . Moreover,

$$(6.18) \quad \frac{d}{dt} \log T_{\text{an}}(h, t) = \text{F.p.}_{s=0} \frac{1}{\Gamma(s)} \int_0^1 x^{s-1} \sum_{q=0}^d (-1)^q \text{tr}_N \left(h e^{-x\Delta_q(t)} (\text{Id} - Q_q(0, t)) \right) dx.$$

Next, $\text{tr}_N (h e^{-x\Delta_q(t)} (\text{Id} - Q_q(0, t))) = \text{tr}_N (h e^{-x\Delta_q^+(t)}) - \text{tr}_N (h Q_q(0, t))$. Further,

$$\text{F.p.}_{s=0} \frac{1}{\Gamma(s)} \int_0^1 x^{s-1} dx = \text{F.p.}_{s=0} \frac{s}{\Gamma(s+1)} \frac{1}{s} = 1$$

and the heat kernel expansion for the Schwartz kernel $K_q(y, y', x, t)$ of $e^{-x\Delta_q(t)}$ on the diagonal $y = y'$ is of the form

$$(6.19) \quad K_q(y, y', x, t) = \sum_{j=0}^{N-1} x^{\frac{j-d}{2}} l_j(y, t) + O(x^{\frac{N-d}{2}}, t)$$

where $l_j(y, t)$ are densities defined on M with values in \mathcal{B} . By a standard parity argument one concludes that $l_d(\cdot, t) = 0$ and argues as in the classical case to conclude that

$$(6.20) \quad \text{F.p.}_{s=0} \frac{1}{\Gamma(s)} \int_0^1 x^{s-1} \sum_{q=0}^d (-1)^q \text{tr}_N (h e^{-x\Delta_q(t)} (\text{Id} - Q_q(0, t))) dx \\ = \sum_{q=0}^d (-1)^{q+1} \text{tr}_N (h Q_q(0, t)) = \sum_{q=0}^d (-1)^{q+1} \text{tr}_N (Q_q(0, t) h Q_q(0, t)).$$

We have proved the following

Proposition 6.2. $\frac{d}{dt} \log T_{\text{an}}(h, t) = \sum_{q=0}^d (-1)^{q+1} \text{tr}_N (Q_q(0, t) h Q_q(0, t)).$

Next, we express the terms $\text{tr}_N (Q_q(0, t) h Q_q(0, t))$ in a more explicit way. It is convenient to introduce $P_q(t) = Q_q(0, t)$. Consider $K_q(t) : \mathcal{H}_t^q(M; \mathcal{E}) \longrightarrow \mathcal{H}_q(M; \mathcal{E})$ defined by

$$(6.21) \quad K_q(t)(\omega) := P_q(0) e^{th} \omega.$$

Using the decomposition $(\omega \in \mathcal{H}_q(M; \mathcal{E})) \ e^{-th} \omega = e^{-th} \omega_+(t) + \omega_0(t) \in \Lambda_t^{+,q}(M; \mathcal{E}) \oplus \mathcal{H}_t^q(M; \mathcal{E})$ where $\omega_+(t) \in \Lambda_t^{+,q}(M; \mathcal{E})$ and $\omega_0(t) \in \mathcal{H}_t^q(M; \mathcal{E})$, one verifies that $P_q(t) e^{-th}$ is the right inverse of $K_q(t)$. Therefore, $K_q(t)$ is an isomorphism. This implies that $K'_q(t) = (K_q(t) K_q(t)^*)^{\frac{1}{2}}$ is a selfadjoint, positive, \mathcal{A} -linear operator on $\mathcal{H}_q(M; \mathcal{E})$ and thus admits a determinant with $\det_N K'_q(t) > 0$. Note that $K_q(t)^*$ is given by $P_q(t) e^{th}$ and thus $K_q(t) K_q(t)^*$ can be written as

$$(6.22) \quad K_q(t) K_q(t)^* = P_q(0) e^{th} P_q(t) e^{th} P_q(0).$$

Lemma 6.3. $\text{tr}_N (P_q(t) h P_q(t)) = \frac{d}{dt} \log \det_N (K_q(t) K_q(t)^*)^{\frac{1}{2}}.$

Proof Using Proposition 1.9 we note that

$$(6.23) \quad \frac{d}{dt} \log \det_N (K_q(t) K_q(t)^*)^{\frac{1}{2}} = \frac{1}{2} \frac{d}{dt} \log \det_N (K_q(t) K_q(t)^*) \\ = \frac{1}{2} \text{tr}_N \left(\frac{d}{dt} (K_q(t) K_q(t)^*) (K_q(t) K_q(t)^*)^{-1} \right).$$

Using (6.22) and writing $\dot{P}_q(t) = \frac{d}{dt} P_q(t)$, we obtain

$$(6.24) \quad \frac{d}{dt} (K_q(t) K_q(t)^*) = P_q(0) h e^{th} P_q(t) e^{th} P_q(0) + P_q(0) e^{th} \dot{P}_q(t) e^{th} P_q(0) + \\ + P_q(0) e^{th} P_q(t) h e^{th} P_q(0).$$

To compute $\dot{P}_q(t) = \frac{d}{dt}(P_q(t)^2) = \dot{P}_q(t)P_q(t) + P_q(t)\dot{P}_q(t)$ we consider the orthogonal decomposition $\Lambda^q(M; \mathcal{E}) = \mathcal{H}_t^q(M; \mathcal{E}) \oplus \Lambda_t^{+,q}(M; \mathcal{E}) \oplus \Lambda_t^{-,q}(M; \mathcal{E})$. An element $\omega \in \Lambda^q(M; \mathcal{E})$ can be uniquely written as

$$(6.25) \quad \omega = \omega_0(t) + e^{-th}\omega_+(t) + e^{th}\omega_-(t)$$

where $\omega_{\pm}(t) \in \Lambda^{\pm,q}(M; \mathcal{E})$ and $\omega_0(t) = P_q(t)\omega(t)$. We conclude that

$$(6.26) \quad \begin{aligned} 0 &= \frac{d}{dt}\omega = \dot{\omega}_0(t) + e^{-th}\dot{\omega}_+(t) + e^{th}\dot{\omega}_-(t) \\ &\quad - he^{-th}\omega_+(t) + he^{th}\omega_-(t). \end{aligned}$$

Note that $\dot{\omega}_{\pm}(t) \in \Lambda^{\pm,q}(M; \mathcal{E})$ and therefore $e^{-th}\dot{\omega}_+(t) \in \Lambda_t^{+,q}(M; \mathcal{E})$ and $e^{th}\dot{\omega}_-(t) \in \Lambda_t^{-,q}(M; \mathcal{E})$. Applying $P_q(t)$ to (6.26) leads to

$$0 = P_q(t)\dot{\omega}_0(t) - P_q(t)he^{-th}\omega_+(t) + P_q(t)he^{th}\omega_-(t).$$

Denoting by $P_q^{\pm}(t)$ the orthogonal projectors $\Lambda^q(M; \mathcal{E}) \longrightarrow \Lambda_t^{\pm,q}(M; \mathcal{E})$ we therefore obtain

$$(6.27) \quad P_q(t)\dot{P}_q(t) = P_q(t)hP_q^+(t) - P_q(t)hP_q^-(t).$$

Observe that the projectors $P_q(t)$ and therefore $\dot{P}_q(t)$ are selfadjoint to conclude that

$$(6.28) \quad \dot{P}_q(t)P_q(t) = P_q^+(t)hP_q(t) - P_q^-(t)hP_q(t).$$

Combining (6.27) and (6.28) we obtain

$$(6.29) \quad \begin{aligned} \dot{P}_q(t) &= \dot{P}_q(t)P_q(t) + P_q(t)\dot{P}_q(t) \\ &= P_q(t)hP_q^+(t) + P_q^+(t)hP_q(t) - P_q(t)hP_q^-(t) - P_q^-(t)hP_q(t). \end{aligned}$$

We apply formula (6.29) to rewrite (6.24),

$$(6.30) \quad \begin{aligned} P_q(0)e^{th}\dot{P}_q(t)e^{th}P_q(0) &= P_q(0)e^{th}P_q(t)hP_q^+(t)e^{th}P_q(0) + P_q(0)e^{th}P_q^+(t)hP_q(t)e^{th}P_q(0) \\ &\quad - P_q(0)e^{th}P_q(t)hP_q^-(t)e^{th}P_q(0) - P_q(0)e^{th}P_q^-(t)hP_q(t)e^{th}P_q(0). \end{aligned}$$

To simplify (6.30), notice that $d_q(t)^* = e^{th}d_q^*e^{-th}$ and therefore $e^{th}\mathcal{H}_q(M; \mathcal{E}) \subset \mathcal{H}_t^q(M; \mathcal{E}) \oplus \Lambda_t^{-,q}(M; \mathcal{E})$ which implies that $P_q^+(t)e^{th}P_q(0) = 0$. Taking the adjoint, we conclude that $P_q(0)e^{th}P_q^+(t) = 0$. Thus, the first two terms on the right hand side of (6.30) are zero. Moreover $P_q^-(t)e^{th}P_q(0) = (\text{Id} - P_q(t))e^{th}P_q(0)$ as well as $P_q(0)e^{th}P_q^-(t) = P_q(0)e^{th}(\text{Id} - P_q(t))$. Applying these considerations to (6.24) yields

$$(6.31) \quad \begin{aligned} \frac{d}{dt}(K_q(t)K_q(t)^*) &= P_q(0)he^{th}P_q(t)e^{th}P_q(0) - P_q(0)e^{th}P_q(t)h(\text{Id} - P_q(t))e^{th}P_q(0) \\ &\quad + P_q(0)e^{th}P_q(t)he^{th}P_q(0) - P_q(0)e^{th}(\text{Id} - P_q(t))hP_q(t)e^{th}P_q(0) \\ &= 2P_q(0)e^{th}P_q(t)hP_q(t)e^{th}P_q(0) \\ &= 2K_q(t)P_q(t)hP_q(t)K_q(t)^*. \end{aligned}$$

Substituting (6.31) into (6.23) leads to

$$\begin{aligned}
\frac{d}{dt} \log \det_N (K_q(t) K_q(t)^*)^{\frac{1}{2}} &= \text{tr}_N (K_q(t) P_q(t) h P_q(t) K_q(t)^*) (K_q(t) K_q(t)^*)^{-1}) \\
&= \text{tr}_N (K_q(t) P_q(t) h P_q(t) K_q(t)^{-1}) \\
&= \text{tr}_N (P_q(t) h P_q(t))
\end{aligned}$$

which concludes the proof of the lemma. \diamond

Using that $K_q(0) = \text{Id}$ and therefore that $\det_N (K_q(0) K_q(0)^*)^{\frac{1}{2}} = 1$, Proposition 6.2 together with Lemma 6.3 lead to

$$\begin{aligned}
\log T_{\text{an}}(h, t) &= \log T_{\text{an}}(h, 0) + \sum_{q=0}^d (-1)^{q+1} \int_0^t \frac{d}{dt} \log \det_N (K_q(t) K_q(t)^*)^{\frac{1}{2}} dt \\
(6.32) \quad &= \log T_{\text{an}}(h, 0) + \sum_{q=0}^d (-1)^{q+1} \log \det_N (K_q(t) K_q(t)^*)^{\frac{1}{2}}.
\end{aligned}$$

In section 4.1 we introduced the \mathcal{A} -linear isomorphisms

$$\theta_q : \text{Null} \Delta_q^{\text{comb}} \longrightarrow \mathcal{H}_q(M; \mathcal{E})$$

and the metric part of the Reidemeister torsion $T_{\text{met}}(\tau)$ defined by

$$\log T_{\text{met}}(M, g, \mathcal{W}, \tau) = \frac{1}{2} \sum_{q=0}^d (-1)^q \log \det_N (\theta_q^* \theta_q).$$

By applying the analysis of the Witten deformation of the deRham complex by Helffer-Sjöstrand (cf section 5) we obtain

Lemma 6.4. *For t sufficiently large, the following statements hold:*

$$\begin{aligned}
\log \det_N (K_q(t) K_q(t)^*)^{\frac{1}{2}} &= \log \det_N (\theta_q^* \theta_q)^{\frac{1}{2}} \\
(6.33) \quad &+ q \beta_q t + \beta_q \left(\frac{d-2q}{4} \right) \log \left(\frac{t}{\pi} \right) + O(t^{-1})
\end{aligned}$$

and

$$\begin{aligned}
\sum_{q=0}^d (-1)^{q+1} \log \det_N (K_q(t) K_q(t)^*)^{\frac{1}{2}} &= -\log T_{\text{met}}(M, g, \mathcal{W}, \tau) + \sum_{q=0}^d (-1)^{q+1} q \beta_q t \\
(6.34) \quad &+ \sum_{q=0}^d (-1)^{q+1} \beta_q \frac{d-2q}{4} \log \left(\frac{t}{\pi} \right) + O(t^{-1}).
\end{aligned}$$

Proof Summing with respect to q , statement (6.34) follows directly from statement (6.33) and the definition $\log T_{\text{met}}(M, g, \mathcal{W}, \tau) = \frac{1}{2} \sum_{q=0}^d (-1)^q \log \det_N(\theta_q^* \theta_q)$. To prove (6.33) we use Stokes' theorem to write

$$K_q(t) = \theta_q K'_q$$

where $K'_q(t) = \pi_q K''_q(t) I_q(t)$, the map $\pi_q : \mathcal{C}^q \longrightarrow \text{Null} \Delta_q^{\text{comb}}$ denotes orthogonal projection, $I_q(t) : \mathcal{H}_t^q(M; \mathcal{E}) \longrightarrow \Lambda^q(M; \mathcal{E})_{\text{sm}}$ denotes inclusion and

$$K''_q(t) : \Lambda^q(M; \mathcal{E})_{\text{sm}} \longrightarrow \mathcal{C}^q, \quad \omega(t) \longrightarrow \text{Int}^{(q)}(e^{th} \omega(t)).$$

Note that

$$(6.35) \quad \log \det_N(K_q(t) K_q(t)^*)^{\frac{1}{2}} = \log \det_N(\theta_q^* \theta_q)^{\frac{1}{2}} + \frac{1}{2} \log \det_N(K'_q(t) K'_q(t)^*).$$

To compute $\log \det_N(K'_q(t) K'_q(t)^*)$ introduce the scaled version of $K''_q(t)$,

$$(6.36) \quad K'''_q(t) := \left(\frac{\pi}{t}\right)^{\frac{(d-2q)}{4}} e^{-tq} K''_q(t)$$

Then, with $K''''_q(t) := \pi_q(t) K'''_q(t) I_q(t)$,

$$(6.37) \quad \log \det_N(K'_q(t) K'_q(t)^*) = \beta_q \frac{(d-2q)}{4} \log \left(\frac{t}{\pi}\right) + \beta_q 2qt + \log \det_N(K''''_q(t) K''''_q(t)^*).$$

In view of Corollary 5.8, we can apply Proposition 1.17 to the map $K''''_q(t)$ and therefore obtain

$$(6.38) \quad \log \det_N(K''''_q(t) K''''_q(t)^*) = O(t^{-1}).$$

Formula (6.33) now follows from (6.35), (6.37) and (6.38).

Lemma 6.5. *For $t \longrightarrow \infty$,*

$$\begin{aligned} \log T_{\text{sm}}(h, t) &= \log T_{\text{comb}}(\tau) \\ &+ \frac{1}{2} \left(\sum_{q=0}^d (-1)^{q+1} q (\beta_q - m_q l) \right) (2t - \log t + \log \pi) + o(1). \end{aligned}$$

Proof Recall that $\log T_{\text{sm}}(h, t)$ is a real number defined by

$$(6.46) \quad \log T_{\text{sm}}(h, t) = \frac{1}{2} \left(\sum_{q=0}^d (-1)^{q+1} q \log \det_N \Delta_q(t)_{\text{sm}} \right)$$

and that, for any $0 < C < \infty$,

$$(6.47) \quad \begin{aligned} \log \det_{\mathbb{N}} \Delta_q(t)_{\text{sm}} &= - \left. \frac{\partial}{\partial s} \right|_{s=0} \frac{1}{\Gamma(s)} \int_0^C \frac{dx}{x} x^s \left(\text{tr}_{\mathbb{N}} \left(e^{-x \Delta_q(t)_{\text{sm}}} \right) - \beta_q \right) \\ &\quad - \int_C^\infty \frac{dx}{x} \left(\text{tr}_{\mathbb{N}} \left(e^{-x \Delta_q(t)_{\text{sm}}} \right) - \beta_q \right) \end{aligned}$$

where $\beta_q = \dim_{\mathbb{N}} \text{Null} \Delta_q^{\text{comb}} = \dim_{\mathbb{N}} \text{Null} \Delta_q(t)$. In view of Theorem 5.4, (5), introduce $\tilde{\Delta}_q(t)_{\text{sm}} := \frac{\pi}{t} e^{2t} \Delta_q(t)_{\text{sm}}$. By a change of variable of integration, $y = \frac{t}{\pi} e^{-2t} x$, we obtain

$$\begin{aligned} \int_{C \frac{\pi}{t} e^{2t}}^\infty \frac{dx}{x} \left(\text{tr}_{\mathbb{N}} \left(e^{-x \Delta_q(t)_{\text{sm}}} \right) - \beta_q \right) &= \int_C^\infty \frac{dy}{y} \left(\text{tr}_{\mathbb{N}} \left(e^{-y \tilde{\Delta}_q(t)_{\text{sm}}} \right) - \beta_q \right) \\ &= \int_{0+}^\infty \left(\int_C^\infty \frac{1}{y} e^{-\mu y} dy \right) dF_{\tilde{\Delta}_q(t)_{\text{sm}}}(\mu) \\ &= \int_{0+}^\infty \left(\int_C^\infty e^{-\mu y} dy \right) F_{\tilde{\Delta}_q(t)_{\text{sm}}}(\mu) d\mu \end{aligned}$$

where $F_{\tilde{\Delta}_q(t)_{\text{sm}}}$ has been introduced in (1.9) and the last equality follows from integration by parts. From Theorem 5.7, (5) and Proposition 1.18 we conclude that there exists $t_0 > 0$ such that, for $t \geq t_0$ and $0 \leq q \leq d$, $F_{\tilde{\Delta}_q(t)_{\text{sm}}}(\mu) \leq F_{\Delta_q^{\text{comb}}}(10\mu)$. For $t \geq t_0$ and $0 \leq q \leq d$, the above computations lead to

$$(6.48) \quad \int_{C \frac{\pi}{t} e^{2t}}^\infty \frac{dx}{x} \left(\text{tr}_{\mathbb{N}} \left(e^{-x \Delta_q(t)_{\text{sm}}} \right) - \beta_q \right) \leq \int_{\frac{C}{10}}^\infty \frac{dx}{x} \left(\text{tr}_{\mathbb{N}} \left(e^{-x \Delta_q^{\text{comb}}} \right) - \beta_q \right).$$

Taking into account that (M, \mathcal{W}) is of determinant class, one can choose $C > 0$ such that

$$(6.49) \quad \int_{\frac{C}{10}}^\infty \frac{dx}{x} \left(\text{tr}_{\mathbb{N}} \left(e^{-x \Delta_q^{\text{comb}}} \right) - \beta_q \right) \leq \epsilon.$$

Therefore, for all $t \geq t_0$, $0 \leq q \leq l$, it suffices to consider

$$(6.50) \quad - \left. \frac{\partial}{\partial s} \right|_{s=0} \frac{1}{\Gamma(s)} \int_0^{C \frac{\pi}{t} e^{2t}} \frac{dx}{x} x^s \left(\text{tr}_{\mathbb{N}} \left(e^{-x \Delta_q(t)_{\text{sm}}} \right) - \beta_q \right).$$

By Theorem 5.7, $\Delta_q(t)_{\text{sm}}$, when expressed in convenient coordinates, is of the form $\frac{t}{\pi} e^{-2t} (\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))$. By a change of variable of integration, $y = \frac{t}{\pi} e^{-2t} x$, the expression (6.50) can be written as a sum of two terms, $I_q + II_q$, where

$$\begin{aligned} I_q &= - (\log \pi - \log t + 2t) III_q(s) \Big|_{s=0} \\ II_q &= - \left. \frac{d}{ds} \right|_{s=0} III_q(s) \end{aligned}$$

and

$$III_q(s) = \frac{1}{\Gamma(s)} \int_0^C \frac{dy}{y} y^s \left(\text{tr}_N \left(e^{-y(\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))} \right) - \beta_q \right).$$

We first evaluate $III_q(s)$ at $s = 0$. Use $\frac{1}{\Gamma(s)} = \frac{s}{\Gamma(s+1)}$ and integrate by parts to obtain, for arbitrary $\delta > 0$,

$$\begin{aligned} (6.51) \quad III_q(s)|_{s=0} &= \frac{1}{\Gamma(s+1)} \int_0^\delta dy \frac{d}{dy} (y^s) \left(\text{tr}_N \left(e^{-y(\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))} \right) - \beta_q \right) \Big|_{s=0} \\ &= \text{tr}_N \left(e^{-\delta(\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))} \right) - \beta_q \\ &\quad + \int_0^\delta dy \text{tr}_N \left((\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}})) e^{-y(\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))} \right). \end{aligned}$$

Taking the limit as $\delta \rightarrow 0$ in (6.51) leads to

$$(6.52) \quad III_q(s)|_{s=0} = m_q l - \beta_q.$$

To compute $II_q = -\frac{d}{ds} \Big|_{s=0} III_q(s)$ recall that

$$\begin{aligned} \log \det_N \Delta_q^{\text{comb}} &= - \frac{\partial}{\partial s} \Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^C \frac{dy}{y} y^s \left(\text{tr}_N \left(e^{-y(\Delta_q^{\text{comb}})} \right) - \beta_q \right) \\ &\quad - \int_C^\infty \frac{dy}{y} \left(\text{tr}_N \left(e^{-y(\Delta_q^{\text{comb}})} \right) - \beta_q \right) \end{aligned}$$

and use the estimate ($0 \leq y \leq C$)

$$(6.53) \quad \left| \text{tr}_N \left(e^{-y(\Delta_q^{\text{comb}})} \right) - \text{tr}_N \left(e^{-y(\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))} \right) \right| \leq y O \left(t^{-\frac{1}{2}} \right)$$

to conclude, together with (6.49) that

$$(6.54) \quad |II_q - \log \det_N \Delta_q^{\text{comb}}| \leq \epsilon + O \left(t^{-\frac{1}{2}} \right).$$

Combining (6.46)-(6.48) and (6.52)-(6.54) we conclude that for given $\epsilon > 0$, there exists $t_\epsilon > 0$ so that for all $t > t_\epsilon$,

$$\left| \log T_{\text{sm}}(h, t) - \log T_{\text{comb}}(\tau) - \frac{1}{2} \sum_{q=0}^d (-1)^{q+1} q (\beta_q - m_q l) (2t - \log \frac{t}{\pi}) \right| \leq 3\epsilon. \quad \diamond$$

Proof of Theorem A First note that $\log T_{\text{la}}(h, t) = \log T_{\text{an}}(h, t) - \log T_{\text{sm}}(h, t)$. Therefore the asymptotic expansion of $\log T_{\text{la}}(h, t)$ is obtained from the expansions of $\log T_{\text{an}}(h, t)$ and $\log T_{\text{sm}}(h, t)$. The asymptotic expansion for $\log T_{\text{an}}(h, t)$ follows from (6.32) and Lemma 6.4 together with the fact that, since d is odd, $\chi(M, \tau) = \sum_{q=0}^d (-1)^q \beta_q = 0$. The asymptotic expansion for $\log T_{\text{sm}}(h, t)$ is contained in Lemma 6.5.

6.2 Comparison theorem for Witten's deformation of the analytic torsion.

The family of operators $\Delta_q(t)$ is a family with parameter of order 2 and weight 1 (cf [Sh1] and section 3). The family $\Delta_q(t)$ fails to be elliptic with parameter precisely at the critical points of the Morse function h . We can therefore use the Mayer-Vietoris type formula for determinants (cf section 3) to localize the failure of the family $\Delta_q(t)$ to be elliptic with parameter and to obtain a relative result which compares the asymptotic expansions of Witten's deformation of the analytic torsion corresponding to two different systems (M^d, h, g, \mathcal{W}) and $(M'^d, h', g', \mathcal{W})$ where the manifolds M and M' have the same fundamental group Γ and (h, g) and (h', g') are generalized triangulations.

Theorem B. *Let d be odd. Suppose that $\tau = (h, g)$ and $\tau' = (h', g')$ are generalized triangulations with $\#\text{Cr}_q(h) = \#\text{Cr}_q(h')$, $(0 \leq q \leq d)$, and that (M, \mathcal{W}) and (M', \mathcal{W}) are of determinant class. Then the following statements hold:*

- (1) *The free term $\text{FT}(\log T_{\text{la}}(h, t) - \log T_{\text{la}}(h', t))$ of the asymptotic expansion of $\log T_{\text{la}}(h, t) - \log T_{\text{la}}(h', t)$ is given by*

$$(6.55) \quad \text{FT}(\log T_{\text{la}}(h, t) - \log T_{\text{la}}(h', t)) \\ = \int_{M \setminus \text{Cr}(h)} a_0(h, \epsilon = 0) - \int_{M' \setminus \text{Cr}(h')} a_0(h', \epsilon = 0)$$

where the densities $a_0(h, \epsilon = 0)$ and $a_0(h', \epsilon = 0)$ are smooth forms of degree d and are given by explicit local formulas and the difference appearing in the right hand side of (6.55) is taken in the sense explained in the remark below, (6.56).

- (2) *Due to the assumption that d is odd,*

$$a_0(h, \epsilon = 0, x) + a_0(d - h, \epsilon = 0, x) = 0.$$

Remark The integral $\int_{M \setminus \text{Cr}(h)} a_0(h, \epsilon = 0)$ need not be convergent and the difference on the right hand side of (6.55) should be understood in the following sense: In view of the definition of a generalized triangulation, there exist neighborhoods V of $\text{Cr}(h)$ and V' of $\text{Cr}(h')$ and a smooth bundle isomorphism $F : \mathcal{E}|_V \longrightarrow \mathcal{E}'|_{V'}$, so that f and F intertwine the functions h and h' , the metrics g and g' and the Laplace operators Δ_q and Δ'_q . Define

$$(6.56) \quad \int_{M \setminus \text{Cr}(h)} a_0(h, \epsilon = 0) - \int_{M' \setminus \text{Cr}(h')} a_0(h', \epsilon = 0) \\ = \int_{M \setminus V} a_0(h, \epsilon = 0) - \int_{M' \setminus V'} a_0(h', \epsilon = 0).$$

Clearly, the definition is independent of the choice of V and V' .

As an application of Theorem A and Theorem B we obtain the following result:

Corollary C. *Let M and M' be two closed manifolds with the same fundamental group and the same dimension d and let \mathcal{W} be an $(\mathcal{A}, \Gamma^{op})$ - Hilbert module of finite type. Suppose that $\tau = (h, g)$ and $\tau' = (h', g')$ are generalized triangulations with $\#\text{Cr}_q(h) = \#\text{Cr}_q(h')$, $(0 \leq q \leq d)$, and that (M, \mathcal{W}) and (M', \mathcal{W}) are of determinant class. Let $T'_{\text{an}} = T_{\text{an}}(M', g', \mathcal{W})$ and $T'_{\text{Re}} = T_{\text{Re}}(M', \tau', \mathcal{W})$. Then*

$$\log T_{\text{an}} - \log T'_{\text{an}} = \log T_{\text{Re}}(\tau) - \log T_{\text{Re}}(\tau').$$

Let $(M, \tau = (h, g))$ be a manifold equipped with a generalized triangulation.

Let $x_{q;j} \in \text{Cr}_q(h)$ be a critical point of h of index q and U_{qj} a system of H -neighbourhoods of $x_{q;j}$ (cf Definition 5.1). Introduce the manifolds

$$M_I := M \setminus \cup_{q,j} U'_{qj}; \quad M_{II} := \cup_{q,j} \overline{U'_{qj}},$$

where U'_{qj} is defined as in Definition 5.1. Both manifolds M_I and M_{II} have the same boundary, given by a disjoint union of spheres of dimension $d - 1$.

Fix $\varepsilon > 0$ and consider the operator $\Delta_q(t) + \varepsilon$. Its symbol with respect to arbitrary coordinates (φ, ψ) of $(M, \mathcal{E} \rightarrow M)$ is of the form

$$(6.57) \quad a_2(x, \xi) + t^2 \|\nabla h\|^2 + a_1(x, \xi) + tL(x) + \epsilon$$

where $a_i : B_{2\alpha} \times \mathbf{R}^d \rightarrow \text{End}(\Lambda^q(\mathbf{R}^d) \otimes \mathcal{W})$ ($i = 1, 2$) are homogeneous of degree i in ξ , where $\|\nabla h\|^2 : B_{2\alpha} \rightarrow \mathbf{R}$ is given by

$$\|\nabla h\|^2 = \sum_{1 \leq i, j \leq d} g^{ij} \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j}$$

and where $L : B_{2\alpha} \rightarrow \text{End}(\Lambda^q(\mathbf{R}^d))$ is the operator $L = \mathcal{L}_{\nabla h} + \mathcal{L}_{\nabla h}^*$ of order 0 with $\mathcal{L}_{\nabla h}$ denoting the Lie-derivative of q -forms along the vector field

$$\nabla h = \sum_{i,j} g^{ij} \frac{\partial h}{\partial x_i} \frac{\partial}{\partial x_j}.$$

The operator $\mathcal{L}_{\nabla h}^*$ is the adjoint of $\mathcal{L}_{\nabla h}$ with respect to the metric g and is given by

$$(6.58) \quad \mathcal{L}_{\nabla h}^* = -(-1)^{q(d+q)} R_{d-q} \mathcal{L}_{\nabla h} R_q$$

where $R_q : \Lambda^q(B_{2\alpha}) \rightarrow \Lambda^{d-q}(B_{2\alpha})$ is the Hodge operator associated to the metric φ^*g . Recall that we have denoted by $\text{Cr}(h)$ the set of all critical points of h . Set $M^* := M \setminus \text{Cr}(h)$. For an arbitrary chart (φ, ψ) of $(M^*, \mathcal{E}|_{M^*} \rightarrow M^*)$, define, as discussed in section 3, the symbol expansion $\sum_{j \geq 0} r_{-2-j}(h, \varepsilon, x, \xi, t, \mu)$ of the resolvent $(\mu - \Delta_q(t) - \varepsilon)^{-1}$ inductively:

$$r_{-2}(h, \varepsilon, x, \xi, t, \mu) = (\mu - a_2(x, \xi) - t^2 \|\nabla h\|^2)^{-1}$$

and, for $j \geq 1$,

$$\begin{aligned}
(6.59) \quad r_{-2-j} &= (\mu - a_2 - t^2 \|\nabla h\|^2)^{-1} \sum_{\substack{1 \leq |\alpha| \leq 2 \\ l+|\alpha|=j}} \frac{1}{\alpha!} \partial_\xi^\alpha a_2 (D_x)^\alpha r_{-2-l} \\
&\quad + (\mu - a_2 - t^2 \|\nabla h\|^2)^{-1} \sum_{\substack{0 \leq |\alpha| \leq 1 \\ l+|\alpha|=j}} \partial_\xi^\alpha (a_1 + tL) (D_x)^\alpha r_{-2-l} \\
&\quad + (\mu - a_2 - t^2 \|\nabla h\|^2)^{-1} \varepsilon r_{-j}.
\end{aligned}$$

Note that r_{-2-j} has the following homogeneity property: for $\lambda \in \mathbf{R}_+$

$$(6.60) \quad r_{-2-j}(h, \varepsilon, x, \lambda \xi, \lambda t, \lambda^{1/2} \mu) = \lambda^{-2-j} r_{-2-j}(h, \varepsilon, x, \xi, t, \mu).$$

For later use, we introduce the densities $a_0(h, \varepsilon, x)$ on M^* with values in \mathbb{R} , defined with respect to the chart (φ, ψ) and arbitrary ε as

$$\begin{aligned}
(6.61) \quad a_0(h, \varepsilon, x) &= \frac{\partial}{\partial s} \Big|_{s=0} \left(\frac{1}{2\pi} \right)^d \frac{1}{2\pi i} \int_{\mathbf{R}^d} d\xi \\
&\quad \int_{\Gamma} d\mu \mu^{-s} \text{tr}_N r_{-2-d}(h, \varepsilon, x, \xi, t=1, \mu) \\
&= \frac{-1}{(2\pi)^d} \int_{\mathbf{R}^d} d\xi \int_0^\infty d\mu \text{tr}_N (r_{-2-d}(h, \varepsilon, x, \xi, t=1, -\mu)).
\end{aligned}$$

Proposition 6.6. *Assume that $(M^d, \tau = (h, g), \mathcal{W})$ and $(M'^d, \tau' = (h', g'), \mathcal{W})$ satisfy the hypothesis of Theorem B. Then for any $\varepsilon > 0$*

(i) *$\log \det_N(\Delta_q(h, t) + \varepsilon) - \log \det_N(\Delta_q(h', t) + \varepsilon)$ has a complete asymptotic expansion for $t \rightarrow \infty$ whose free term is denoted by $\bar{a}_0 := \bar{a}_0(h, h', \varepsilon)$.*

(ii) *The coefficient \bar{a}_0 can be represented in the form*

$$(6.62) \quad \bar{a}_0 = \int_{M_I} a_0(h, \varepsilon, x) - \int_{M'_I} a_0(h', \varepsilon', x')$$

where $a_0(h, \varepsilon, x)$ and $a_0(h', \varepsilon, x')$ are the densities introduced in (6.75) for arbitrary ε .

(iii) *If $\dim M = d$ is odd then*

$$(6.63) \quad \bar{a}_0(h, h', \varepsilon) + \bar{a}_0(d - h, d - h', \varepsilon) = 0 \quad (\text{all } \varepsilon > 0).$$

Proof. The proof is based on a Mayer-Vietoris type formula (Theorem 3.6). Note that $\Delta_q(h, t) + \varepsilon$ is a family of invertible, selfadjoint, elliptic operators with parameter t of order 2 and weight 1 for any $\varepsilon > 0$. The same is true for the operators $(\Delta_q^I(h, t) + \varepsilon)_D$

and $(\Delta_q^{II}(h, t) + \varepsilon)_D$ obtained by restricting $\Delta_q(h, t) + \varepsilon$ to M_I and M_{II} , respectively, and imposing Dirichlet boundary conditions. We can therefore apply Theorem 3.6. Denote by $R_{DN}(h, t, \varepsilon)$ the Dirichlet to Neumann operator defined in section 3.3. We conclude from Theorem 3.6 (4) that $R_{DN}(h, t, \varepsilon)$ is an invertible pseudodifferential operator with parameter of order 1 and weight 2 and from Theorem 3.6 (2) we conclude that $R_{DN}(h, t, \varepsilon)$ is elliptic with parameter t . According to Theorem 3.4, $\log \det_N R_{DN}(h, t, \varepsilon)$ has an asymptotic expansion for $t \rightarrow \infty$. Inspecting the principal symbol of $(\Delta_q^I(h, t) + \varepsilon)_D$ one observes that $(\Delta_q^I(h, t) + \varepsilon)_D$ is a family of invertible, selfadjoint differential operators with parameter of order 2 and weight 1 which is elliptic with parameter. From Theorem 3.5 we therefore conclude that $\log \det_N(\Delta_q^I(h, t) + \varepsilon)_D$ admits a complete asymptotic expansion as $t \rightarrow \infty$. Finally $(\Delta_q^{II}(h, t) + \varepsilon)_D$ is a family of invertible selfadjoint operators with parameter of order 2 and weight 1, which is, however, not elliptic with parameter.

Of course the same considerations can be made for the system (M', h', g') to conclude that $\log \det_N R_{DN}(h', t, \varepsilon)$ and $\log \det_N(\Delta_q^I(h', t) + \varepsilon)_D$ have both asymptotic expansions for $t \rightarrow \infty$. Applying the Mayer-Vietoris type formula (Theorem 3.6 (3)) for $\log \det_N(\Delta_q(h, t) + \varepsilon)$ and $\log \det_N(\Delta_q(h', t) + \varepsilon)$ we obtain for the difference

$$\begin{aligned}
(6.64) \quad & \log \det_N(\Delta_q(h, t) + \varepsilon) - \log \det_N(\Delta_q(h', t) + \varepsilon) \\
&= \log \det_N(\Delta_q^I(h, t) + \varepsilon)_D - \log \det_N(\Delta_q^I(h', t) + \varepsilon)_D \\
&+ \log \det_N(\Delta_q^{II}(h, t) + \varepsilon)_D - \log \det_N(\Delta_q^{II}(h', t) + \varepsilon)_D \\
&+ \log \det_N R_{DN}(h, t, \varepsilon) - \log \det_N R_{DN}(h', t, \varepsilon) \\
&+ \log \bar{c}(h, t, \varepsilon) - \log \bar{c}(h', t, \varepsilon).
\end{aligned}$$

Note that M_{II} and M'_{II} are isometric and $\mathcal{E}_{|M_{II}}$ as well as $\mathcal{E}'_{|M'_{II}}$ are trivial. Consequently

$$\log \det_N(\Delta_q^{II}(h, t) + \varepsilon)_D = \log \det_N(\Delta_q^{II}(h', t) + \varepsilon)_D.$$

Due to our definition of H -coordinates the isometry between M_{II} and \tilde{M}_{II} extends to neighbourhoods of M_{II} and \tilde{M}_{II} . As a consequence we conclude from Theorem 3.6(4) and Theorem 3.4 that $\bar{c}(h, t, \varepsilon) = \bar{c}(h', t, \varepsilon)$ and that $\log \det_N R_{DN}(h, t, \varepsilon)$ and $\log \det_N R_{DN}(h', t, \varepsilon)$ have identical asymptotic expansions. We have therefore proved that

$$\log \det_N(\Delta_q(h, t) + \varepsilon) - \log \det_N(\Delta_q(h', t) + \varepsilon)$$

has an asymptotic expansion as $t \rightarrow \infty$ which is identical to the complete asymptotic expansion for $\log \det_N(\Delta_q^I(h, t) + \varepsilon)_D - \log \det_N(\Delta_q^I(h', t) + \varepsilon)_D$. According to Theorem 3.5 the free term in the asymptotic expansions of both $\log \det_N(\Delta_q^I(h, t) + \varepsilon)_D$ and $\log \det_N(\Delta_q^I(h', t) + \varepsilon)_D$ consists of a boundary contribution and a contribution from the interior. Recall that ∂M_I and $\partial M'_I$ are isometric and that in collar neighbourhoods of ∂M_I and of $\partial M'_I$ the symbols of $(\Delta_q^I(h, t) + \varepsilon)_D$ and $(\Delta_q^I(h', t) + \varepsilon)_D$ are identical when expressed in (H) -coordinates. Therefore the boundary contributions are the same and the

free term in the asymptotic expansion of $\log \det_N(\Delta_q^I(h, t) + \varepsilon) - \log \det_N(\Delta_q^I(h', t) + \varepsilon)$ is given by

$$(6.65) \quad \bar{a}_0 = \int_{M_I} a_0(h, \varepsilon, x) - \int_{M'_I} a_0(h', \varepsilon, x')$$

where the densities $a_0(h, \varepsilon, x)$ and $a_0(h', \varepsilon, x')$ are given by (6.61).

Noting that $a_0(h, \varepsilon, x)$ and $a_0(h', \varepsilon, x')$ are identical on $M_{II} \setminus \text{Cr}(h) \cong M'_{II} \setminus \text{Cr}(h')$ statement (ii) follows. Towards (iii), observe that if M is of odd dimension, the quantity $r_{-d-2}(h, \varepsilon, x, \xi, t, \mu)$ defining $a_0(h, \varepsilon, x)$ satisfies, according to (6.59) and (6.60) ,

$$(6.66) \quad r_{-d-2}(d - h, \varepsilon, x, \xi, t, \mu) = r_{-d-2}(h, \varepsilon, x, \xi, -t, \mu)$$

and

$$(6.67) \quad r_{-d-2}(h, \varepsilon, x, -\xi, -t, \mu) = -r_{-d-2}(h, \varepsilon, x, \xi, t, \mu).$$

Therefore $r_{-d-2}(h, \varepsilon, x, \xi, t, \mu) + r_{-d-2}(d - h, \varepsilon, x, \xi, t, \mu)$ is an odd function of ξ . Integrating over $|\xi| = 1$ we conclude that $a_0(h, \varepsilon, x) + a_0(d - h, \varepsilon, x) = 0$. \square

For any $\varepsilon > 0$ introduce the following perturbed version of $\log T(h, t)$

$$(6.68) \quad A(h, t, \varepsilon) := \frac{1}{2} \sum_{q=0}^d (-1)^{q+1} q \log \det_N(\Delta_q(h, t) + \varepsilon).$$

Note that $A(h, t, \varepsilon)$ can be written as a sum

$$(6.69) \quad A(h, t, \varepsilon) = A_{\text{sm}}(h, t, \varepsilon) + A_{\text{la}}(h, t, \varepsilon)$$

where A_{sm} is defined in a fashion analogous to $\log T_{\text{sm}}(h, t)$,

$$A_{\text{sm}}(h, t, \varepsilon) := \frac{1}{2} \sum_{q=0}^d (-1)^{q+1} q \log \det_N(\Delta_q^{\text{sm}}(h, t) + \varepsilon)$$

with

$$\Delta_q^{\text{sm}}(h, t) := \Delta_q(h, t)|_{\Lambda^q(M; \mathcal{E})_{\text{sm}}}$$

and $A_{\text{la}}(h, t, \varepsilon)$ is given by $A(h, t, \varepsilon) - A_{\text{sm}}(h, t, \varepsilon)$. Observe that the spectrum of the operator $\Delta_q^{\text{sm}}(h, t)$ tends to 0 as $t \rightarrow \infty$ and therefore, by Theorem 5.7 (5)

$$\log \det_N(\Delta_q^{\text{sm}}(h, t) + \varepsilon) = m_q \log \varepsilon + O\left(\frac{1}{\varepsilon} t e^{-2t}\right)$$

for $t \rightarrow \infty$. This shows that $A_{\text{sm}}(h, t, \varepsilon) - A_{\text{sm}}(h', t, \varepsilon)$ is exponentially small as $t \rightarrow \infty$ and hence, for any fixed $\varepsilon > 0$, it has a trivial asymptotic expansion for $t \rightarrow \infty$. In view of (6.69) and Proposition 6.6 we conclude that for any $\varepsilon > 0$, $A(h, t, \varepsilon) - A(h', t, \varepsilon)$ and $A_{\text{la}}(h, t, \varepsilon) - A_{\text{la}}(h', t, \varepsilon)$ have complete asymptotic expansions for $t \rightarrow \infty$ and, moreover, these expansions are identical. In particular we conclude that the free terms of the two expansions are identical

$$FT(A_{\text{la}}(h, t, \varepsilon) - A_{\text{la}}(h', t, \varepsilon)) = FT(A(h, t, \varepsilon) - A(h', t, \varepsilon)).$$

Using Proposition 6.6 (ii) and the fact that the densities $a_0(h, \varepsilon, x)$ and $a_0(h', \varepsilon, x)$ (6.61) are continuous in ε we obtain

Lemma 6.7.

(i) For any $\varepsilon > 0$, $A_{la}(h, t, \varepsilon) - A_{la}(h', t, \varepsilon)$ has a complete asymptotic expansion for $t \rightarrow \infty$ which is identical to the asymptotic expansion for $A(h, t, \varepsilon) - A(h', t, \varepsilon)$.

(ii) The limit

$$\lim_{\varepsilon \rightarrow 0} FT(A_{la}(h, t, \varepsilon) - A_{la}(h', t, \varepsilon))$$

exists and is given by

$$(6.70) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} FT(A_{la}(h, t, \varepsilon) - A_{la}(h', t, \varepsilon)) \\ &= \int_{M_I} a_0(h, \varepsilon = 0, x) - \int_{M'_I} a_0(h', \varepsilon = 0, x'). \end{aligned}$$

We proceed to investigate the left hand side of (6.70). For this we need the following estimate for the spectral function $N_q(t, \lambda)$ of $\Delta_q(t) = \Delta_q(h, t)$.

Lemma 6.8. *There exists a constant $C > 0$ independent of t and $\lambda \geq 1$ such that, for t sufficiently large,*

$$N_q^{la}(t, \lambda) \leq C\lambda^d.$$

Proof. First note that for t sufficiently large, $\Delta_q(t) \geq \Delta_q(0) = \Delta_q$ on M_I . By Weyl's law, we conclude

$$N_q^I(t, \lambda) \leq N_q^I(0, \lambda) \leq C_1 \lambda^{d/2}$$

where $N_q^I(t, \lambda)$ is the spectral function for the operator $\Delta_q(t)$ restricted to M_I with Neumann boundary conditions (Neumann spectrum). Recall that $M_{II} = \cup_{k,j} U_{kj}$. On each of the discs U_{kj} , $\Delta_q(t)$, when expressed in (H)-coordinates, is the direct sum of shifted harmonic oscillators of the form

$$H_t := -\frac{d^2}{dx^2} + t^2 x^2 + tc$$

with $-\alpha < x < \alpha$ (α as in (6.61)). Following [CFKS, p.218] introduce the scaling operator S_t defined by

$$S_t f(x) := t^{1/2} f(tx).$$

Then $S_{t^{1/2}} \cdot tK \cdot S_{t^{1/2}}^{-1} = H_t$ where

$$K := -\frac{d^2}{dx^2} + x^2 + c.$$

Therefore the Neumann spectrum of H_t on the interval $-\alpha < x < \alpha$ is the same as the Neumann spectrum of tK when considered on the interval $-\sqrt{t}\alpha < x < \sqrt{t}\alpha$. Denote by $N_{tK; \sqrt{t}}(\lambda)$ the spectral function of the operator tK on the interval $-\sqrt{t}\alpha < x < \sqrt{t}\alpha$ with Neumann boundary conditions and by $N_{tK; \sqrt{t}}^D(\lambda)$ the spectral function of the operator tK

on the interval $-\sqrt{t}\alpha < x < \sqrt{t}\alpha$ with Dirichlet spectrum. Note that for all $t \geq 0$ and λ sufficiently large

$$N_{tK;\sqrt{t}}(\lambda) \leq N_{tK;\sqrt{t}}^D(\lambda) + 1 \leq 2N_{K;\sqrt{t}}^D(\lambda/t).$$

Comparing the Dirichlet problem for K on $-\sqrt{t}\alpha \leq x \leq \sqrt{t}\alpha$ with the one on the whole real line we conclude that $N_{K;\sqrt{t}}^D(\lambda/t) \leq C_2\lambda/t \leq C_2\lambda$ for $t \geq 1$ with a constant $C_2 > 0$ independent of λ and t . Hence we have shown that the spectral function $N_q^{II}(t, \lambda)$ of the operator $\Delta_q(t)$ on M_{II} with Neumann boundary conditions can be estimated by

$$N_q^{II}(t, \lambda) \leq C_3\lambda^d$$

for a constant $C_3 > 0$ independent of t and λ . The subadditive property of the Neumann spectral function implies that

$$N_q(t, \lambda + 0) \leq N_q^I(t, \lambda + 0) + N_q^{II}(t, \lambda + 0) \leq C\lambda^d$$

for some constant $C > 0$ independent of t and λ and for t sufficiently large. \square

Let us introduce the trace of the heat kernel of $\Delta_q^{\text{la}}(t)$,

$$\theta_q(t, \mu) = \int_1^\infty e^{-\mu\lambda} dN_q^{\text{la}}(t, \lambda).$$

Corollary 6.9.

(i) *There exists a constant $C > 0$ independent of t and μ such that, for t sufficiently large and $\mu > 0$*

$$(6.71) \quad \theta_q(t, \mu) \leq C\mu^{-d}.$$

(ii) *There exist constants $C > 0$ and $\beta > 0$ independent of t and μ such that, for t sufficiently large, and $\mu \geq 1/\sqrt{t}$*

$$(6.72) \quad \theta_q(t, \mu) \leq Ce^{-\beta t\mu}.$$

Proof.

(i) By Proposition 5.2 there exists a constant $C_1 > 0$ such that $\text{spec}(\Delta_q^{\text{la}}(t)) \subset [C_1t, \infty)$ and therefore

$$\theta_q(t, \mu) = \int_{C_1t}^\infty e^{-\mu\lambda} dN_q(t, \lambda).$$

Integrating by parts we obtain

$$(6.73) \quad \theta_q(t, \mu) \leq \mu \int_{C_1t}^\infty e^{-\mu\lambda} N_q(t, \lambda) d\lambda.$$

By Lemma 6.8, one then concludes

$$\theta_q(t, \mu) \leq \frac{C}{\mu^d} \int_{C_1 t \mu}^{\infty} e^{-\lambda} \lambda^d d\lambda \leq \tilde{C}/\mu^d.$$

(ii) From (6.72) and Lemma 6.8 we obtain

$$\theta_q(t, \mu) \leq C\mu e^{-C_1 t \mu/2} \int_{C_1 t}^{\infty} e^{-\mu \lambda/2} \lambda^d d\lambda \leq \frac{\tilde{C}}{\mu^d} e^{-C_1 t \mu/2}.$$

By choosing $\beta < C_1/2$ and $C > 0$ sufficiently large we obtain (ii). \square

Recall from Theorem A that $\log T_{1a}(h, t)$ has an asymptotic expansion for $t \rightarrow \infty$.

Proposition 6.10.

$$\lim_{\varepsilon \rightarrow 0} FT(A_{la}(h, t, \varepsilon) - A_{la}(h', t, \varepsilon)) = FT(\log T_{la}(h, t)) - FT(\log T_{la}(h', t)).$$

Proof. We verify that the function, defined for $\varepsilon > 0$ and t sufficiently large by

$$H(t, \varepsilon) := A_{la}(h, t, \varepsilon) - A_{la}(h', t, \varepsilon) + \log T_{la}(h, t) - \log T_{la}(h', t)$$

is of the form

$$(6.74) \quad H(t, \varepsilon) = \sum_{k=1}^d \varepsilon^k f_k(t) + g(t, \varepsilon)$$

where $g(t, \varepsilon) = o(1)$ uniformly in ε . The statement of the proposition can be deduced from (6.74) as follows: Recall that for $\varepsilon > 0$, $H(t, \varepsilon)$ has an asymptotic expansion for $t \rightarrow \infty$. As $g(t, \varepsilon) = o(1)$ uniformly in ε we conclude that for any $\varepsilon > 0$, $\sum_{k=1}^d \varepsilon^k f_k(t)$ has an asymptotic expansion for $t \rightarrow \infty$. By taking d different values $0 < \varepsilon_1 < \dots < \varepsilon_d$ for ε and using that the Vandermonde determinant is nonzero

$$\det \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_1^d \\ \vdots & & \vdots \\ \varepsilon_d & \dots & \varepsilon_d^d \end{pmatrix} \neq 0$$

we conclude that for any $1 \leq k \leq d$, $f_k(t)$ has an asymptotic expansion for $t \rightarrow \infty$ and that for any $\varepsilon > 0$

$$FT(H(t, \varepsilon)) = \sum_{k=1}^d \varepsilon^k FT(f_k(t)).$$

Hence $\lim_{\varepsilon \rightarrow 0} FT(H(t, \varepsilon))$ exists and $\lim_{\varepsilon \rightarrow 0} FT(H(t, \varepsilon)) = 0$. To prove (6.74) we introduce the zeta function $\zeta_{q,la}$ of $\Delta_q^{la}(t) + \varepsilon$,

$$(6.75) \quad \zeta_{q,la}(t, \varepsilon, s) = \frac{1}{\Gamma(s)} \int_0^\infty \mu^{s-1} \theta_q(t, \mu) e^{-\varepsilon \mu} d\mu$$

with $\theta_q(t, \mu)$ given as above. The integral in (6.75) can be split into two parts

$$(6.76) \quad \zeta_{q,la}^I(t, \varepsilon, s) = \frac{1}{\Gamma(s)} \int_{1/\sqrt{t}}^\infty \mu^{s-1} \theta_q(t, \mu) e^{-\varepsilon \mu} d\mu$$

and

$$(6.77) \quad \zeta_{q,la}^{II}(t, \varepsilon, s) = \frac{1}{\Gamma(s)} \int_0^{1/\sqrt{t}} \mu^{s-1} \theta_q(t, \mu) e^{-\varepsilon \mu} d\mu.$$

First let us consider

$$(6.78) \quad \zeta_{q,la}^I(t, \varepsilon, s) - \zeta_{q,la}^I(t, \varepsilon = 0, s) = \frac{1}{\Gamma(s)} \int_{1/\sqrt{t}}^\infty \mu^s \theta_q(t, \mu) \frac{e^{-\varepsilon \mu} - 1}{\mu} d\mu.$$

Note that

$$\zeta_{q,la}^I(t, \varepsilon, s) - \zeta_{q,la}^I(t, \varepsilon = 0, s)$$

is, by Corollary 6.9 (ii), an entire function of s . Noting that

$$\left. \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \right) \right|_{s=0} = 1$$

and that $1 - e^{-\varepsilon \mu} \leq \varepsilon \mu$, we obtain

$$\begin{aligned} & \left| \frac{d}{ds} \Big|_{s=0} (\zeta_{q,la}^I(t, \varepsilon, s) - \zeta_{q,la}^I(t, \varepsilon = 0, s)) \right| \\ &= \left| \int_{1/\sqrt{t}}^\infty \theta_q(t, \mu) \frac{e^{-\varepsilon \mu} - 1}{\mu} d\mu \right| \\ &\leq \varepsilon C \int_{1/\sqrt{t}}^\infty e^{-\beta t \mu} d\mu = \frac{\varepsilon C}{\beta t} e^{-\beta \sqrt{t}} \end{aligned}$$

where we have used Corollary 6.9. To analyze the term

$$\frac{d}{ds} \Big|_{s=0} (\zeta_{q,la}^{II}(t, \varepsilon, s) - \zeta_{q,la}^{II}(t, \varepsilon = 0, s)),$$

first expand $(e^{-\varepsilon \mu} - 1)/\mu$

$$(e^{-\varepsilon \mu} - 1)/\mu = \sum_{k=1}^d \frac{(-1)^k}{k!} \varepsilon^k \mu^{k-1} + \varepsilon^{d+1} \mu^d e(\varepsilon, \mu)$$

where the error term is given by

$$e(\varepsilon, \mu) = \left(\sum_{k=d+1}^{\infty} \frac{(-1)^k}{k!} \varepsilon^k \mu^{k-1} \right) / \varepsilon^{d+1} \mu^d.$$

Note that, by Corollary 6.9, $\mu^d \theta_q(t, \mu) \leq C$ and therefore

$$\int_0^{1/\sqrt{t}} \mu^s \theta_q(t, \mu) \varepsilon^{d+1} \mu^d e(\varepsilon, \mu) d\mu$$

is a meromorphic function of s , with $s = 0$ a regular point and, for t sufficiently large

$$\left| \frac{d}{ds} \right|_{s=0} \left(\frac{1}{\Gamma(s)} \int_0^{1/\sqrt{t}} \mu^s \theta_q(t, \mu) \varepsilon^{d+1} \mu^d e(\varepsilon, \mu) d\mu \right) \leq \varepsilon^{d+1} C / \sqrt{t}$$

where C is independent of t and ε , $0 \leq \varepsilon \leq 1$. Finally, recall that $\theta_q(t, \mu)$ admits an expansion for $\mu \rightarrow 0+$ of the form

$$\theta_q(t, \mu) = \sum_{j=0}^d C_j(t) \mu^{(j-d)/2} + \theta'_q(t, \mu)$$

where $\theta'_q(t, \mu)$ is continuous in $\mu \geq 0$. Therefore, for $1 \leq k \leq d$,

$$\frac{1}{\Gamma(s)} \int_0^{1/\sqrt{t}} \mu^s \theta_q(t, \mu) \frac{(-1)^k}{k!} \varepsilon^k \mu^{k-1} d\mu$$

is analytic in s at $s = 0$ and

$$\sum_{k=1}^d \frac{d}{ds} \Big|_{s=0} \left(\frac{1}{\Gamma(s)} \int_0^{1/\sqrt{t}} \mu^s \theta_q(t, \mu) \frac{(-1)^k}{k!} \varepsilon^k \mu^{k-1} d\mu \right)$$

is of the form $\sum_{k=1}^d \varepsilon^k f_k(t)$. This establishes (6.74). \square

Proof of Theorem B. From Theorem A we know that $\log T_{la}(h, t) - \log T_{la}(h', t)$ has an asymptotic expansion for $t \rightarrow \infty$. By Proposition 6.10, the free term of the asymptotic expansion \bar{a}_0 is given by

$$\bar{a}_0 = \lim_{\varepsilon \rightarrow 0} FT(A_{la}(h, t, \varepsilon) - A_{la}(h', t, \varepsilon)).$$

By Lemma 6.7 (ii) we conclude that

$$\bar{a}_0 = \int_{M_I} a_0(h, \varepsilon = 0, x) - \int_{M'_I} a_0(h', \varepsilon = 0, x').$$

Equation (6.69) is proved in Proposition 6.5 (iii). In view of the equality

$$FT(\log T_{\text{an}}(h, t) - \log T_{\text{sm}}(h, t)) = FT(\log T_{\text{la}}(h, t))$$

one can see that \bar{a}_0 is independent of h and h' within the class of functions h and h' which give rise to the same cochain complexes $C^*(M; \tau, \mathcal{W})$ respectively $C^*(M; \tau', \mathcal{W}')$. This combined with the locality of a_0 implies that $\int_{M_I} a_0(h, \varepsilon = 0, x) = \int_{M_I} a_0(h', \varepsilon = 0, x')$ which in turn implies (ii). \square

Proof of Corollary C. Choose a bijection $\Theta : \text{Cr}(h) \rightarrow \text{Cr}(h')$ so that $\Theta(x_{q;j})$ is a critical point $x'_{q;j}$ of h' of index q . By assumption Θ extends to an isometry $\Theta : \cup_{q,j} U_{qj} \rightarrow \cup_{q,j} U'_{qj}$ where (U_{qj}) and (U'_{qj}) are systems of H -neighbourhoods for h , respectively h' . Denote by τ , respectively, τ' the triangulation induced by (h, g) , respectively (h', g') and by $\tau_{\mathcal{D}}$, respectively $\tau'_{\mathcal{D}}$ the triangulations $\tau_{\mathcal{D}} = (d - h, g)$, respectively $\tau'_{\mathcal{D}} = (d - h', g')$. It follows from Poincaré duality and d odd that $\log T_{\text{met}}(\tau) = \log T_{\text{met}}(\tau_{\mathcal{D}})$ and $\log T_{\text{comb}}(\tau) = \log T_{\text{comb}}(\tau_{\mathcal{D}})$.

Using Theorem A for both h and $d - h$, we obtain

$$\begin{aligned} 2 \log T_{\text{an}} - 2 \log T'_{\text{an}} &= FT(\log T_{\text{an}}(h, t) - \log T_{\text{an}}(h', t)) \\ &\quad + FT(\log T_{\text{an}}(d - h, t) - \log T_{\text{an}}(d - h', t)) \\ &\quad + 2 \log T_{\text{met}}(\tau) - 2 \log T_{\text{met}}(\tau') \end{aligned}$$

Decomposing $\log T_{\text{an}}(h, t) = \log T_{\text{la}}(h, t) + \log T_{\text{sm}}(h, t)$ and taking into account the asymptotics of $\log T_{\text{sm}}(h, t)$ (cf Theorem A) we conclude that

$$\begin{aligned} 2 \log T_{\text{an}} - 2 \log T'_{\text{an}} &= 2 \log T_{\text{comb}}(\tau) - 2 \log T_{\text{comb}}(\tau') \\ &\quad + FT(\log T_{\text{la}}(h, t) - \log T_{\text{la}}(h', t)) \\ &\quad + FT(\log T_{\text{la}}(d - h, t) - \log T_{\text{la}}(d - h', t)) \end{aligned}$$

from which the Corollary follows by (6.55) and (6.56). \square

6.3 Proof of Theorem 2.

In this subsection we provide the proof of Theorem 2 using Corollary C of subsection 6.2 together with the product formulas for the Reidemeister torsion and the analytic torsion established in Proposition 4.1.

First we need the following result concerning the metric anomaly of the analytic torsion, which is a straightforward generalization of a classical result due to Ray-Singer [RS], and can be proved by arguments similar to the ones presented in subsection 6.1.

Lemma 6.11. *Let M^d be a closed manifold of odd dimension d such that (M, \mathcal{W}) is of determinant class. Assume that $g(u)$ is a smooth one-parameter family of Riemannian*

metrics on M . Then $\log T_{\text{an}}(M, g(u), \mathcal{W})$ is a smooth function of u whose derivative is given by

$$(6.79) \quad \frac{d}{du} \log T_{\text{an}}(M, g(u), \mathcal{W}) = \frac{1}{2} \frac{d}{du} \sum_{q=0}^d (-1)^q \log \det_{\mathbb{N}}(\sigma_q(u)^* \sigma_q(u))$$

where $\sigma_q(u)$ is the \mathcal{A} -linear, bounded isomorphism

$$\sigma_q(u) : \text{Null} \Delta_q(u_0) \longrightarrow \text{Null} \Delta_q(u)$$

provided by Hodge theory and u_0 is arbitrary but fixed.

Given generalized triangulations $\tau = (h, g')$ and $\tau' = (h', g'')$ of M , τ' is called a subdivision of τ if

- (1) $\text{Cr}_q(h) \subset \text{Cr}_q(h')$ ($0 \leq q \leq d$)
- (2) $W_x^\pm(h', g'') \subset W_x^\pm(h, g')$ for any $x \in \text{Cr}_q(h)$.

The following result can be found in [Mi2]:

Lemma 6.12. *Let $\tau = (h, g')$ be a generalized triangulation, $0 \leq q \leq d-1$ an integer and x, y two distinct points in $M \setminus \text{Cr}(h)$. Then there exists a generalized triangulation $\tau' = (h', g'')$ with the following properties:*

- (1) $\text{Cr}_k(h') = \text{Cr}_k(h)$ for $k \neq q, q+1$;
- (2) $\text{Cr}_q(h') = \text{Cr}_q(h) \cup \{x\}$; $\text{Cr}_{q+1}(h') = \text{Cr}_{q+1}(h) \cup \{y\}$;
- (3) τ' is a subdivision of τ ;
- (4) $W_y^- \cap W_x^+$ is connected.

Since the Reidemeister torsion does not change under subdivision (cf [Mi1]), one obtains

$$T_{\text{Re}}(M, g, \mathcal{W}, \tau) = T_{\text{Re}}(M, g, \mathcal{W}, \tau').$$

Proof of Theorem 2 By Lemma 6.12, which concerns the metric anomaly, and in view of the definition of T_{met} , it suffices to prove Theorem 1 in the case where $g = g'$, $\tau = (g', h)$.

Consider the sphere $S^6 = \{x = (x_1, \dots, x_7) \in \mathbb{R}^7; \sum x_j^2 = 1\}$ with an arbitrary generalized triangulation $\tau_1 = (h_1, g_1)$. Let $\tau = (h, g)$ be a generalized triangulation for M and consider $M \times S^6$, endowed with the Riemannian metric $g \times g_1$ and the triangulation $\tau \times \tau_1 = (h + h_1, g \times g_1)$. Note that $\Gamma = \pi_1(M) = \pi_1(M \times S^6)$. By assumption, (M, \mathcal{W}) is of determinant class. As S^6 is of determinant class, $(M \times S^6, \mathcal{W})$ is of determinant class as well. Moreover, by the product formulas of Proposition 4.1,

$$(6.80) \quad \log T_{\text{an}}(M \times S^6, g \times g_1, \mathcal{W}) = 2 \log T_{\text{an}}(M, g, \mathcal{W})$$

and

$$(6.81) \quad \log T_{\text{Re}}(M \times S^6, g \times g_1, \mathcal{W}, \tau \times \tau_1) = 2 \log T_{\text{Re}}(M, g, \mathcal{W}, \tau)$$

where we used that $\chi(S^6) = 2$ and that $\chi(M, \mathcal{W}) = 0$ (as M is of odd dimension).

Next, consider the product $S^3 \times S^3$ of the 3-spheres, $S^3 = \{x = (x_1, \dots, x_4) \in \mathbb{R}^4; \sum x_j^2 = 1\}$, with an arbitrary generalized triangulation $\tau_2 = (h_2, g_2)$. Arguing as above, we conclude that $(M \times S^3 \times S^3, \mathcal{W})$ is of determinant class and that, by the product formulas of Proposition 4.1,

$$\log T_{\text{an}}(M \times S^3 \times S^3, g \times g_2, \mathcal{W}) = 0$$

and

$$\log T_{\text{Re}}(M \times S^3 \times S^3, g \times g_2, \mathcal{W}, \tau \times \tau_2) = 0$$

where we used that $\chi(S^3 \times S^3) = 0$ and that $\chi(M, \mathcal{W}) = 0$.

Choose a subdivision $\tau' = (h', g')$ of the generalized triangulation $\tau \times \tau_1$ in $M \times S^6$ and a subdivision $\tau'' = (h'', g'')$ of the generalized triangulation $\tau \times \tau_2$ in $M \times S^3 \times S^3$ so that, for $0 \leq q \leq 6$, $\#\text{Cr}_q(h') = \#\text{Cr}_q(h'')$. This is possible because $M \times S^6$ and $M \times S^3 \times S^3$ are both of odd dimension and therefore $\chi(M \times S^3 \times S^3, \mathcal{W}) = \chi(M \times S^6, \mathcal{W}) = 0$. We conclude from the above, Corollary C, Lemma 6.11 and Lemma 6.12 that

$$\begin{aligned} & 2 \log T_{\text{an}}(M, g, \mathcal{W}) - 2 \log T_{\text{Re}}(M, g, \mathcal{W}, \tau) \\ &= \log T_{\text{an}}(M \times S^6, g \times g_1, \mathcal{W}) - \log T_{\text{Re}}(M \times S^6, g \times g_1, \mathcal{W}, \tau \times \tau_1) \\ &= \log T_{\text{an}}(M \times S^6, g', \mathcal{W}) - \log T_{\text{Re}}(M \times S^6, g', \mathcal{W}, \tau') \\ &= \log T_{\text{an}}(M \times S^3 \times S^3, g'', \mathcal{W}) - \log T_{\text{Re}}(M \times S^3 \times S^3, g'', \mathcal{W}, \tau'') \\ &= \log T_{\text{an}}(M \times S^3 \times S^3, g \times g_2, \mathcal{W}) - \log T_{\text{Re}}(M \times S^3 \times S^3, g \times g_2, \mathcal{W}, \tau \times \tau_2) = 0. \end{aligned}$$

This proves Theorem 2.

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